Value of Information and Geometry of Optimal Decision-making and Learning

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Optimal learning

First variational problem

Value of information

Iterative algorithms as Markov morphisms

Optimal transition kernels

Optimal control of mutation rate

Optimal learning

Learning, Optimization, Self-Adaptation

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- Is there a difference between Bayesian (statistical) estimation and control theory and theory of learning systems?
- Learning algorithms usually involve iterations $x_{t+1} = \Lambda(x_t)$.
- Learning problems are characterized by prior uncertainty.

What is learning?

Proposition

There is no need to learn, if there is nothing to optimize.

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There is nothing to learn, if one has full information.

Learning Systems

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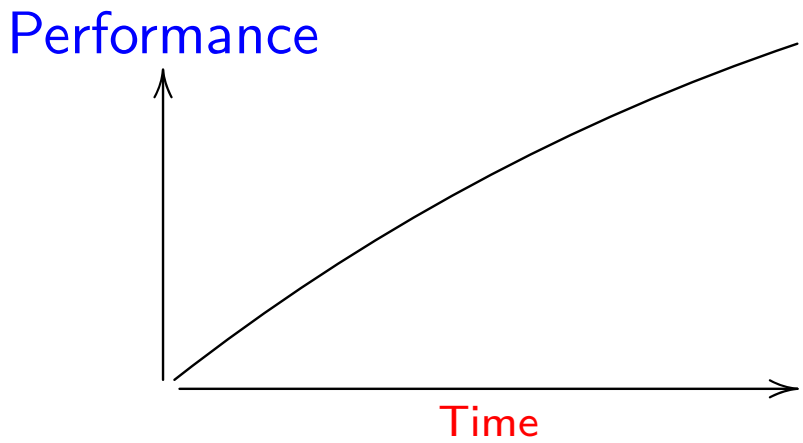

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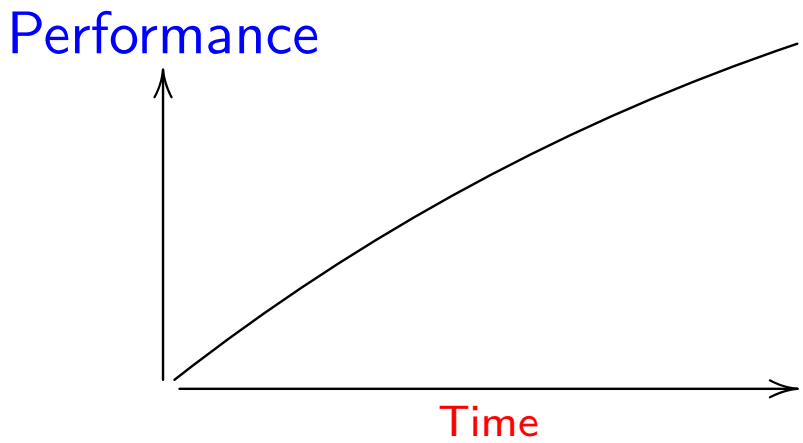

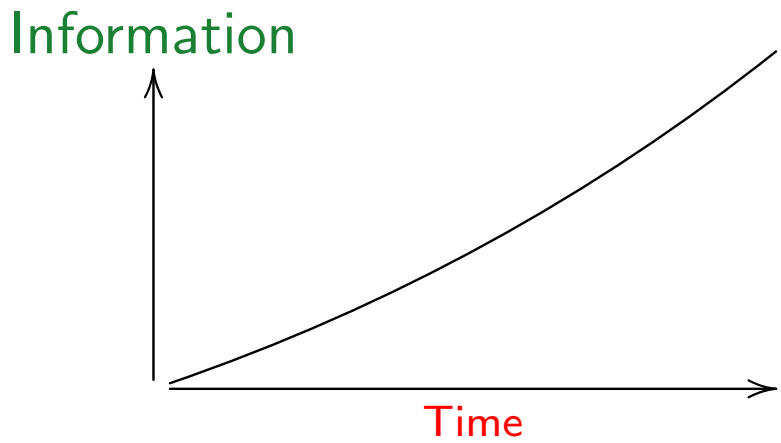

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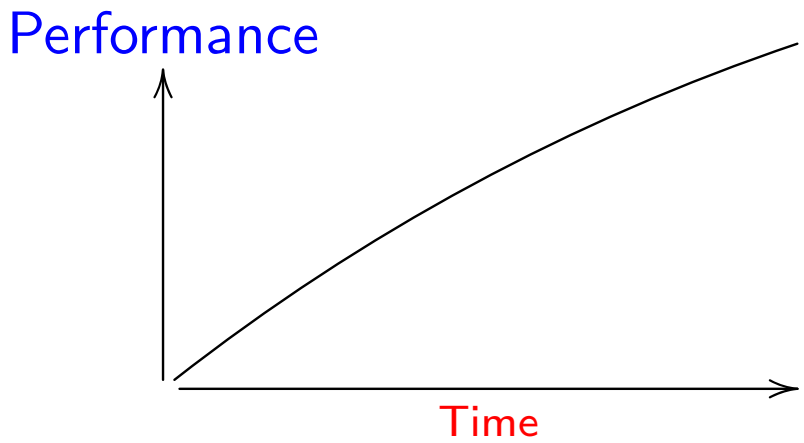
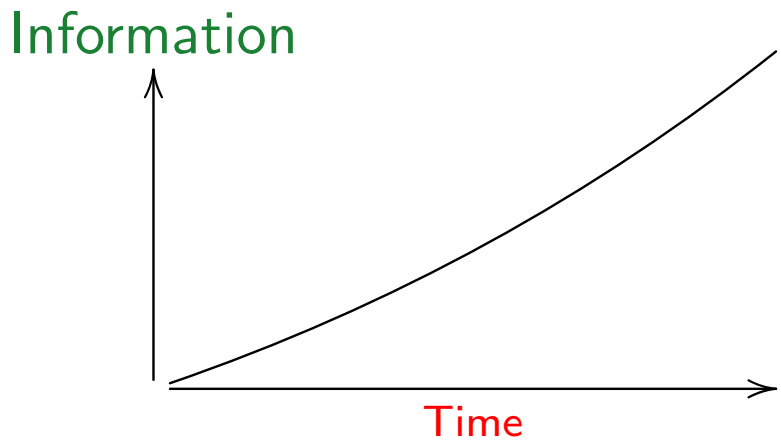
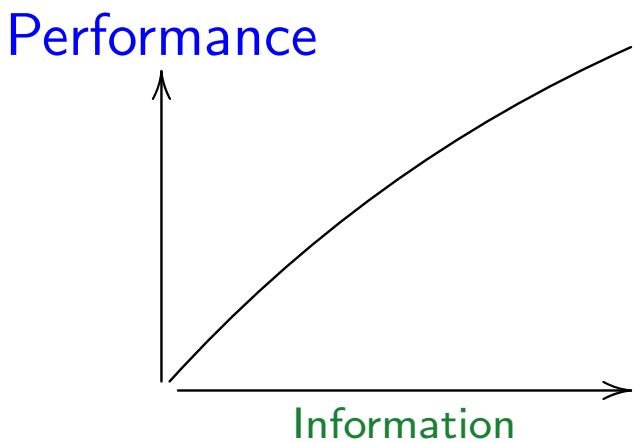




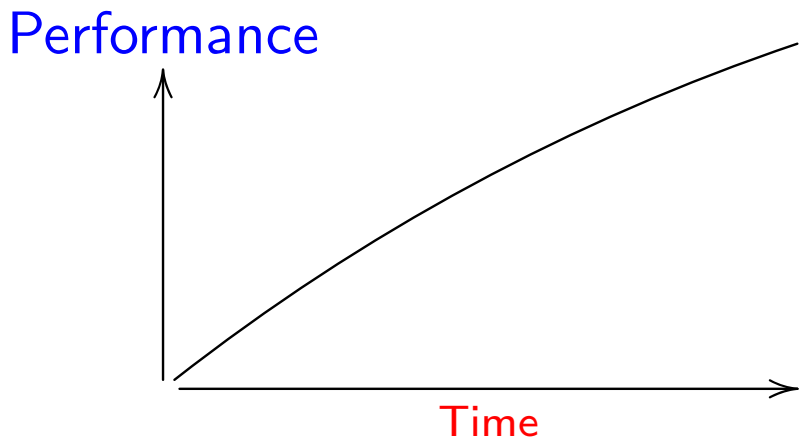
Optimal learning

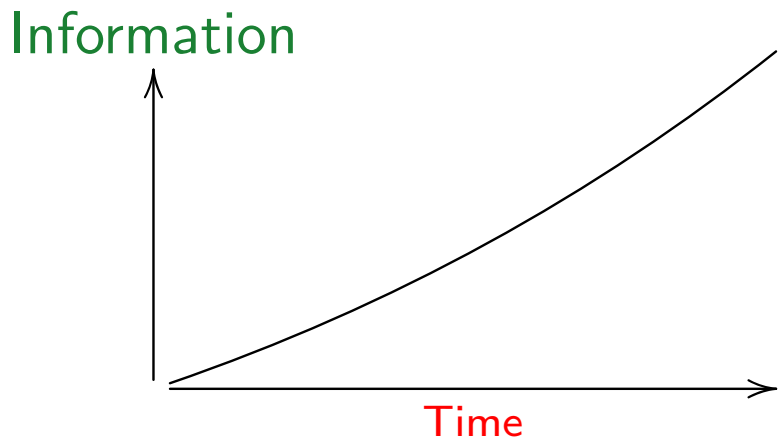


Maximize performance

s.t. information $\leq \lambda$

Learning Systems

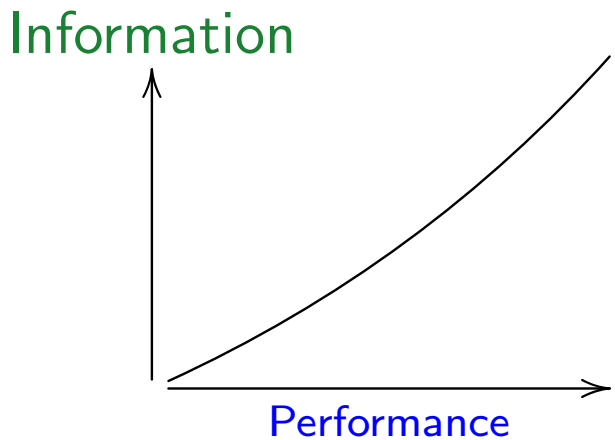




Optimal learning

Minimize information

s.t. performance $\geq v$



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Example (Least squares estimation)

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$$\nabla_z \mathbb{E}_p\{u(x, z)\} = \sum \nabla_z u(x, z)\, p(x) = \hat{z} - \sum x\, p(x) = 0$$

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Dynamics of learning

- Given n i.i.d. observatons of E with probability $p(E)$:

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- Oh! The mice learn the distribution of the cat.

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Information-geometruc view

- The set of all probability measures

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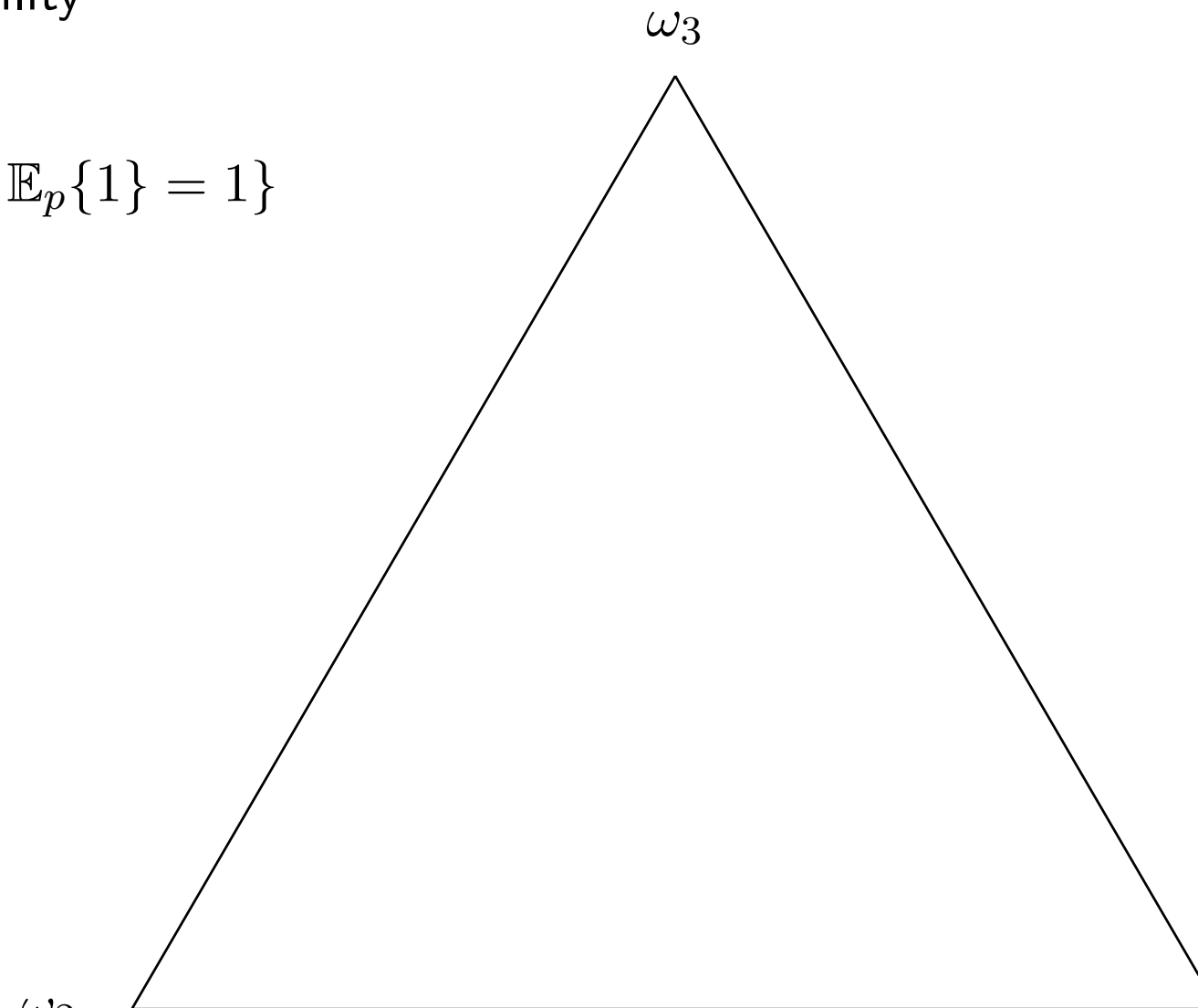

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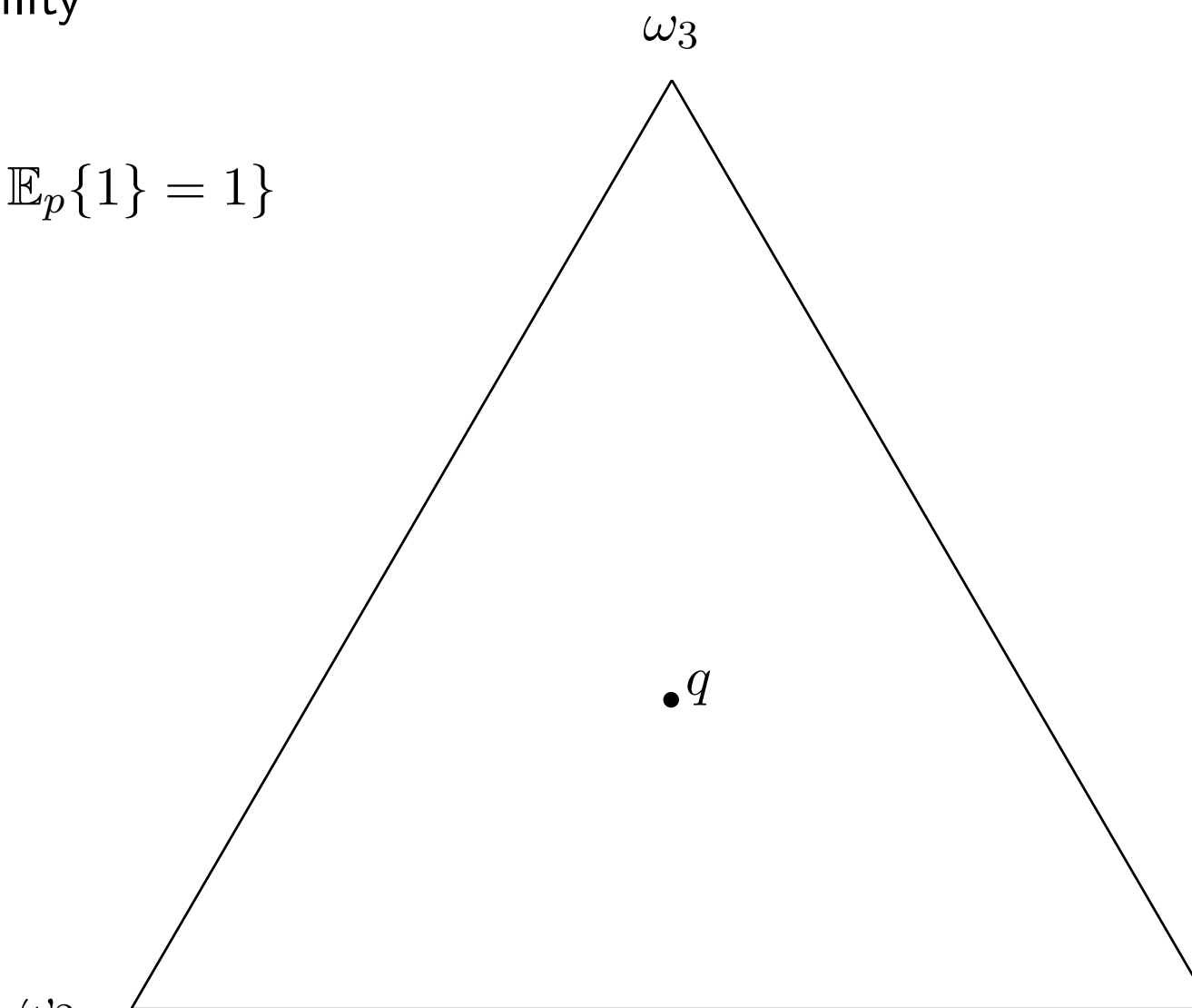

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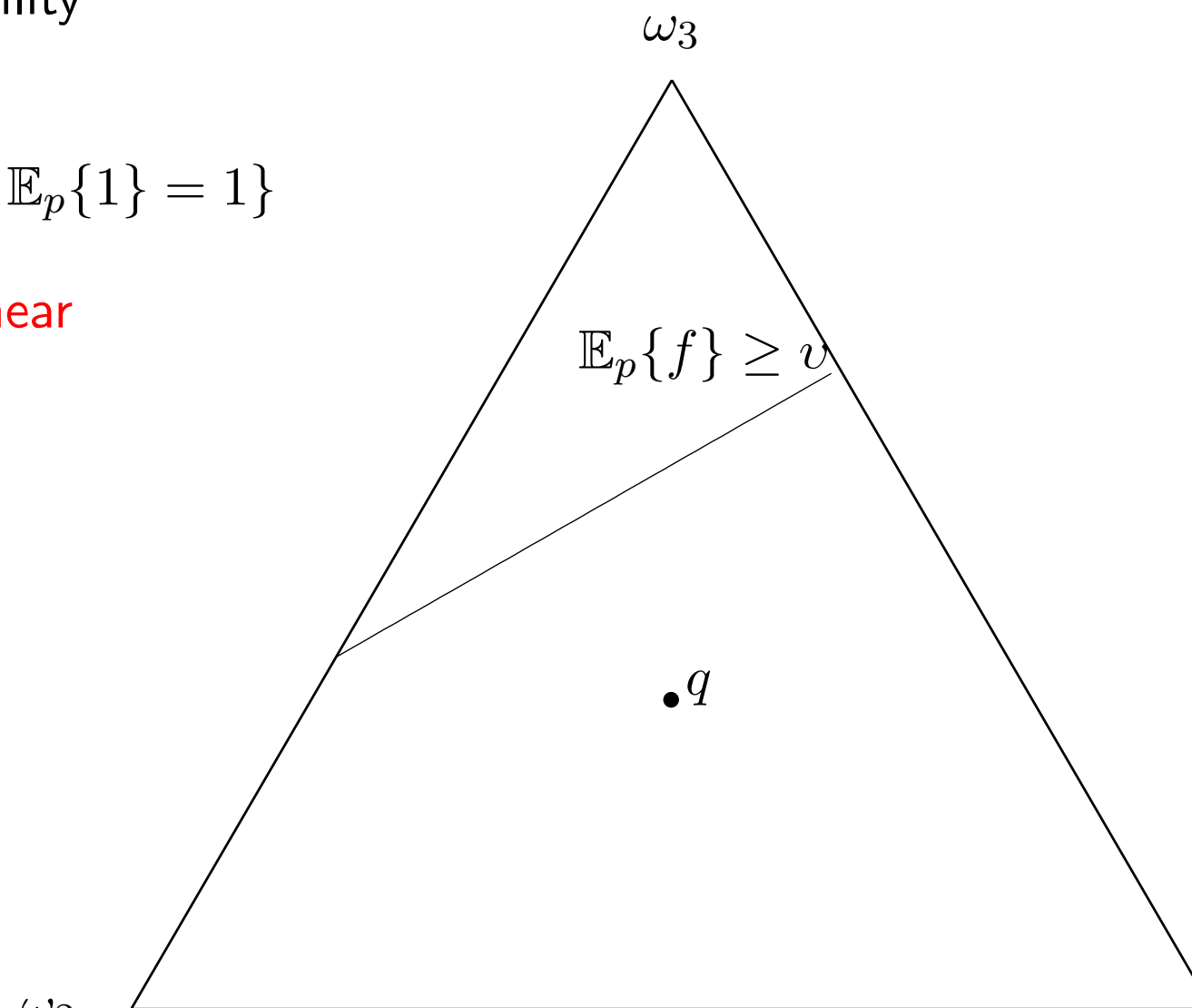

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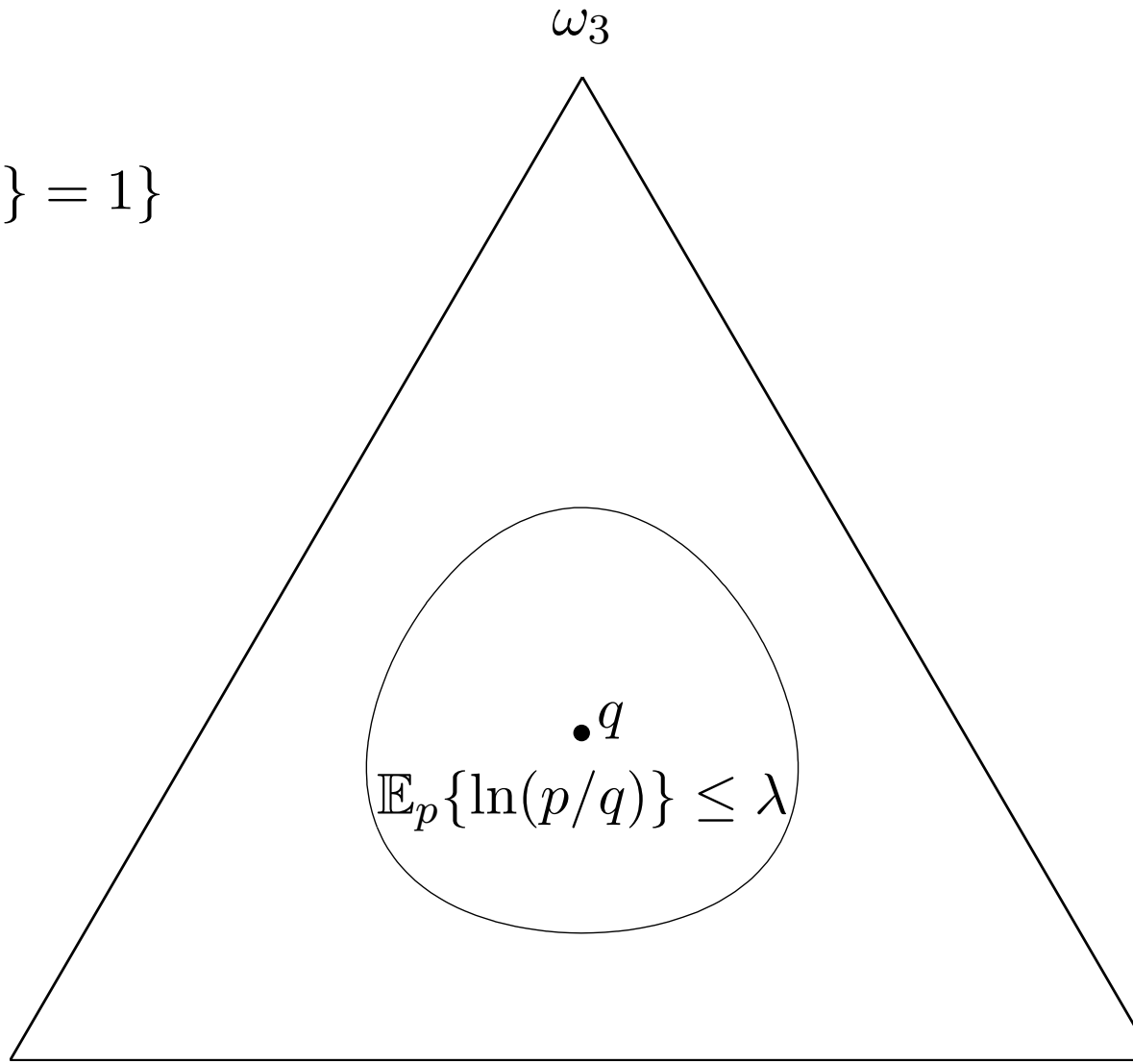

First variational problem

Problem I

- Linear programming problem:

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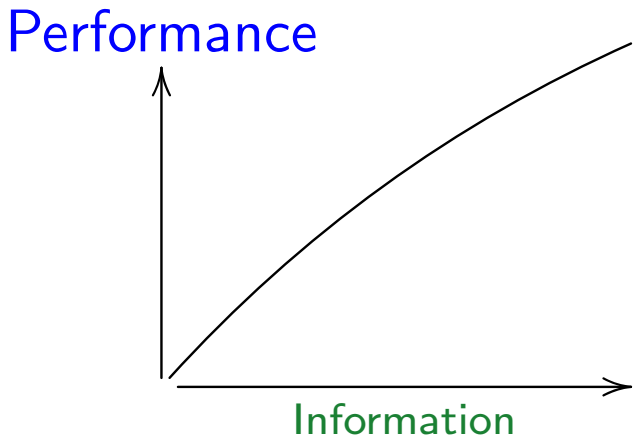

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- The inverse convex programming problem:

$$\text{minimize} \quad \mathbb{E}_p\{\ln(p/q)\} \quad \text{subject to} \quad \mathbb{E}_p\{u\} \ge \upsilon \quad \left(\mathbb{E}_p\{u\} \le \upsilon\right)$$

Minimize information

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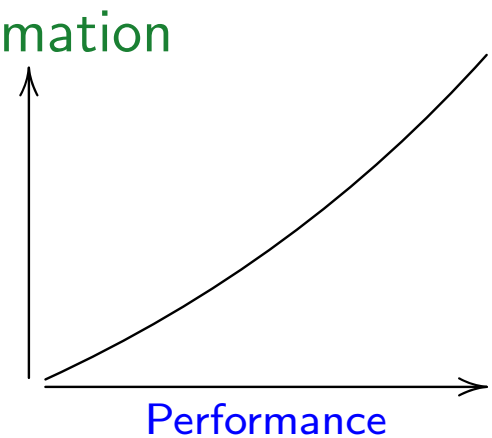

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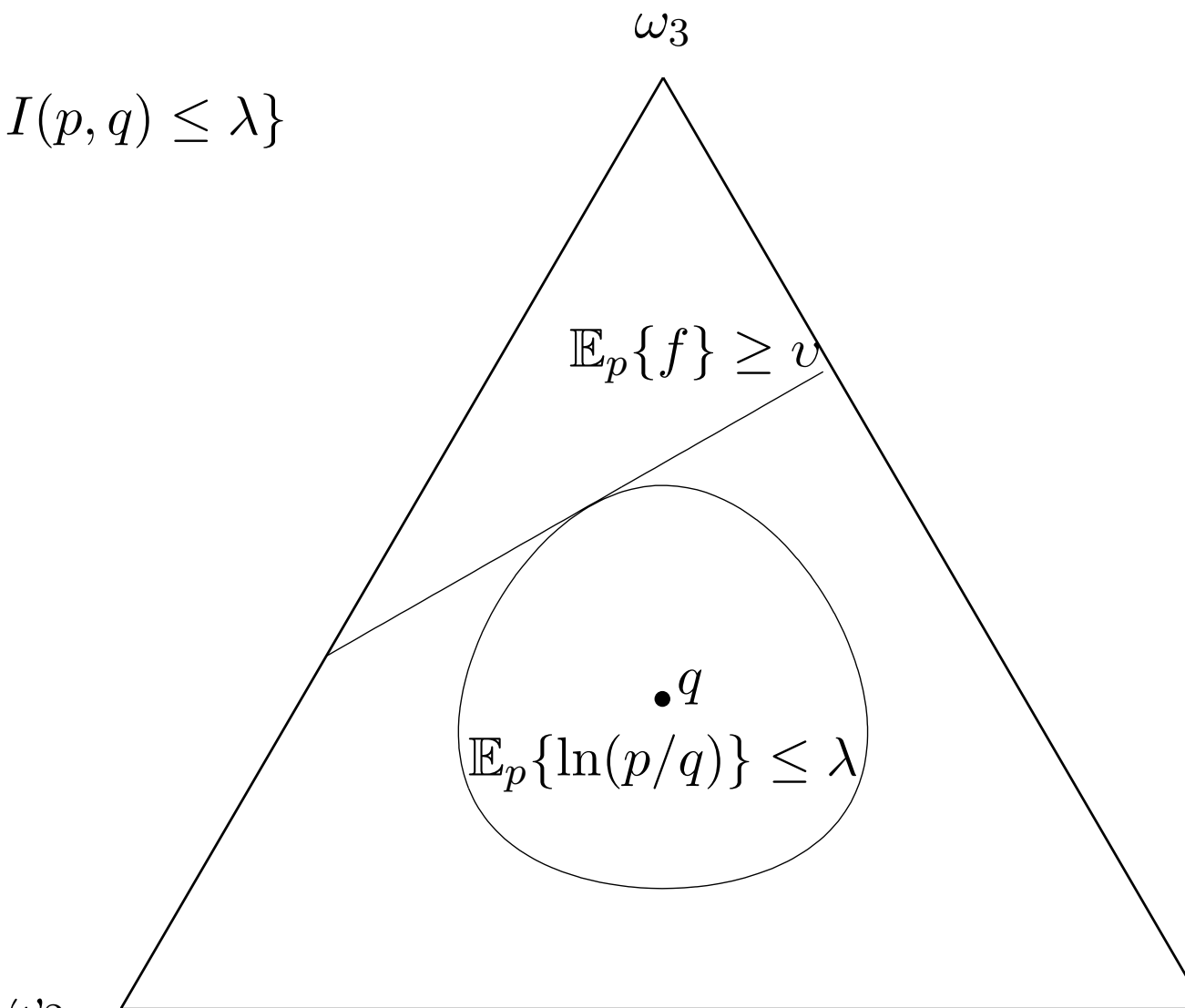

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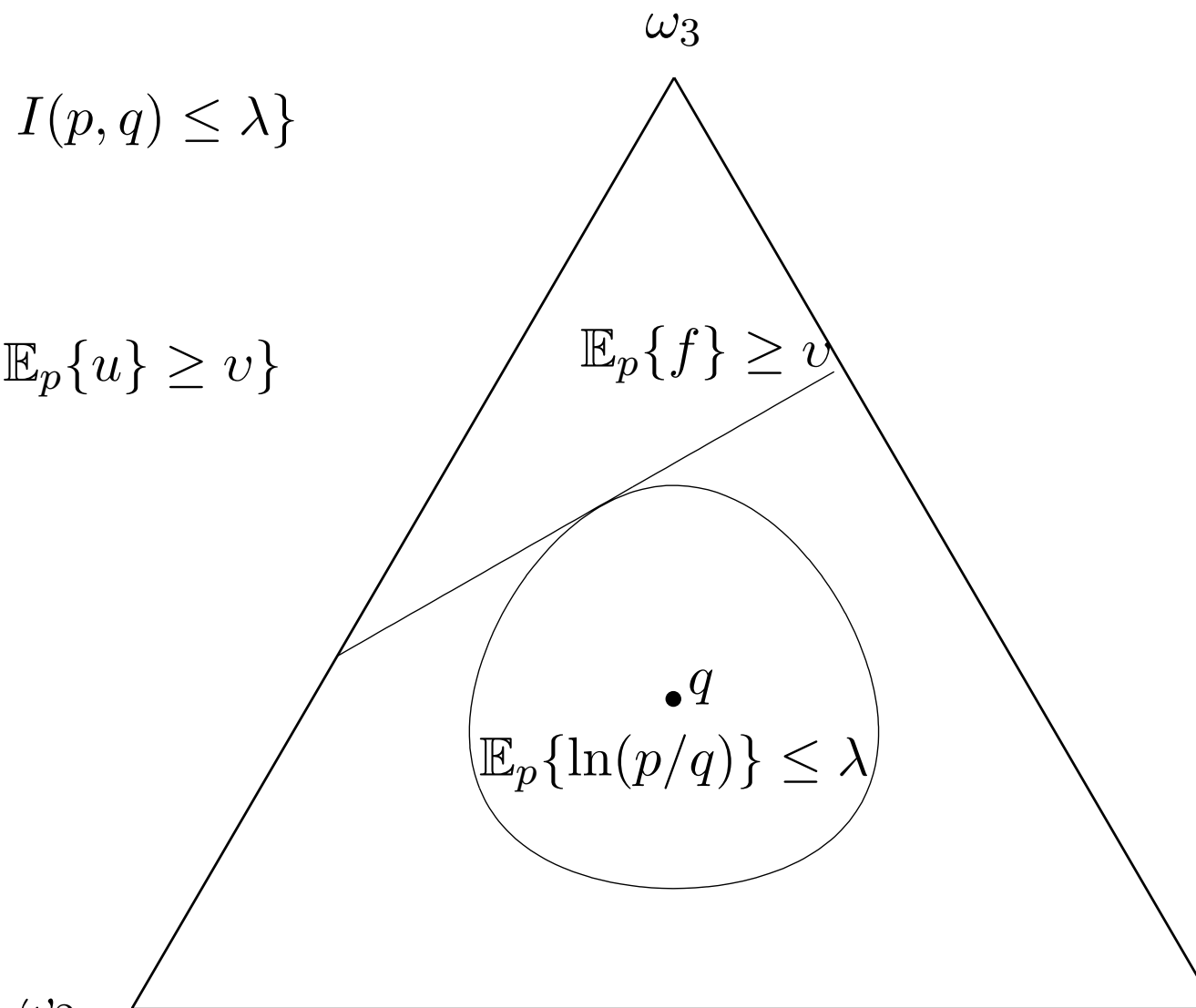

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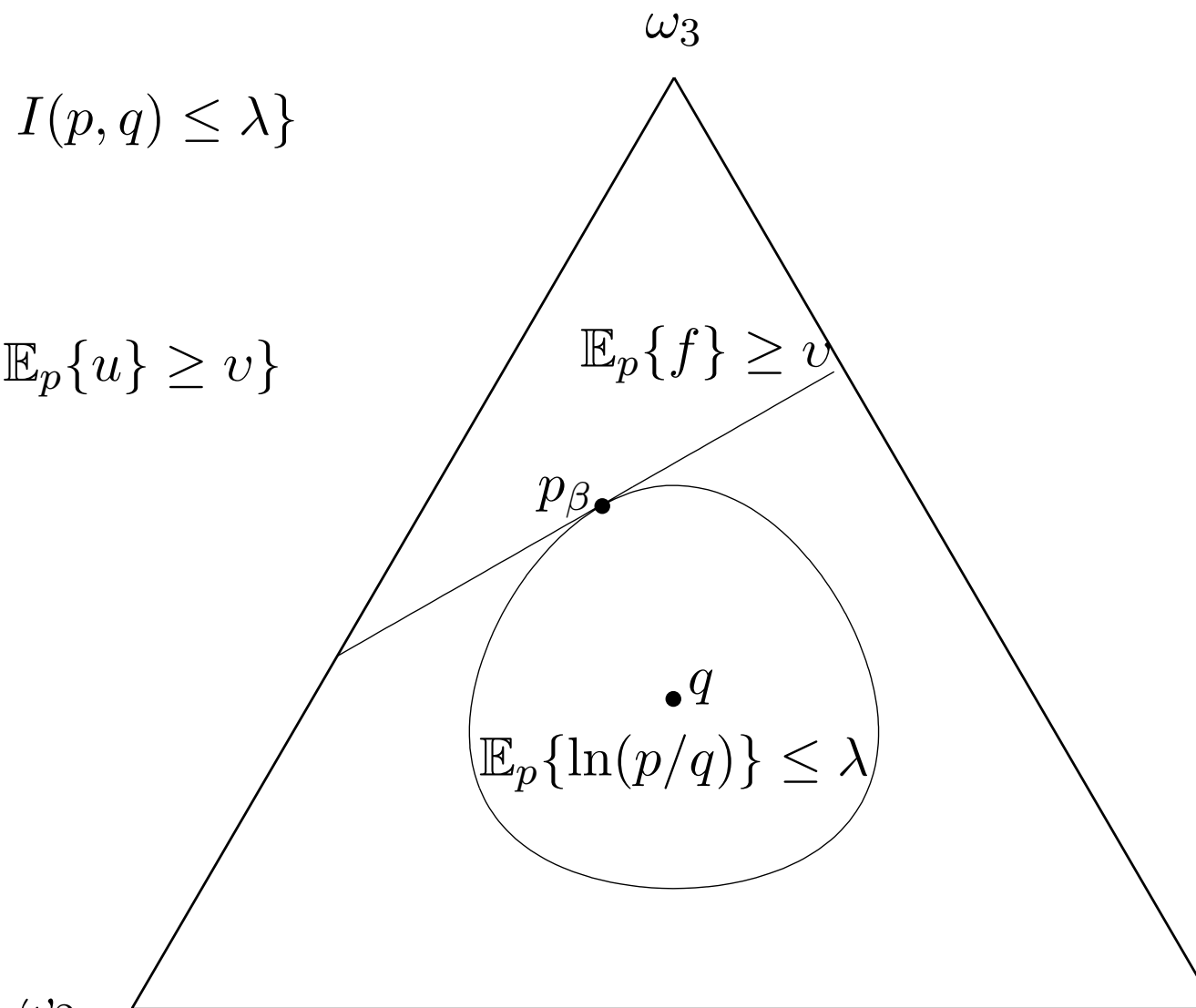

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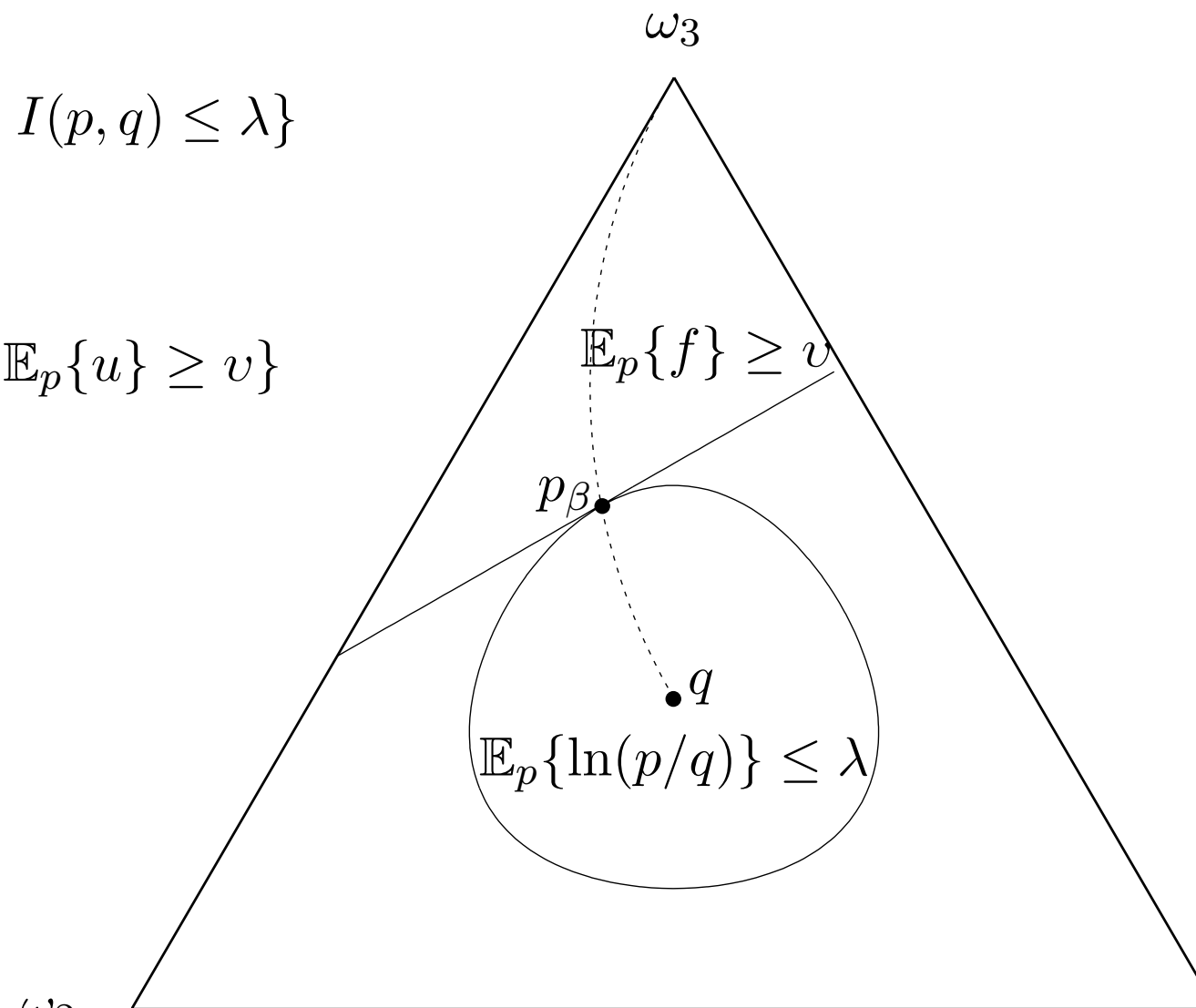

Solution

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- Necessary and sufficient conditions $\nabla K(p, \beta) = 0$:

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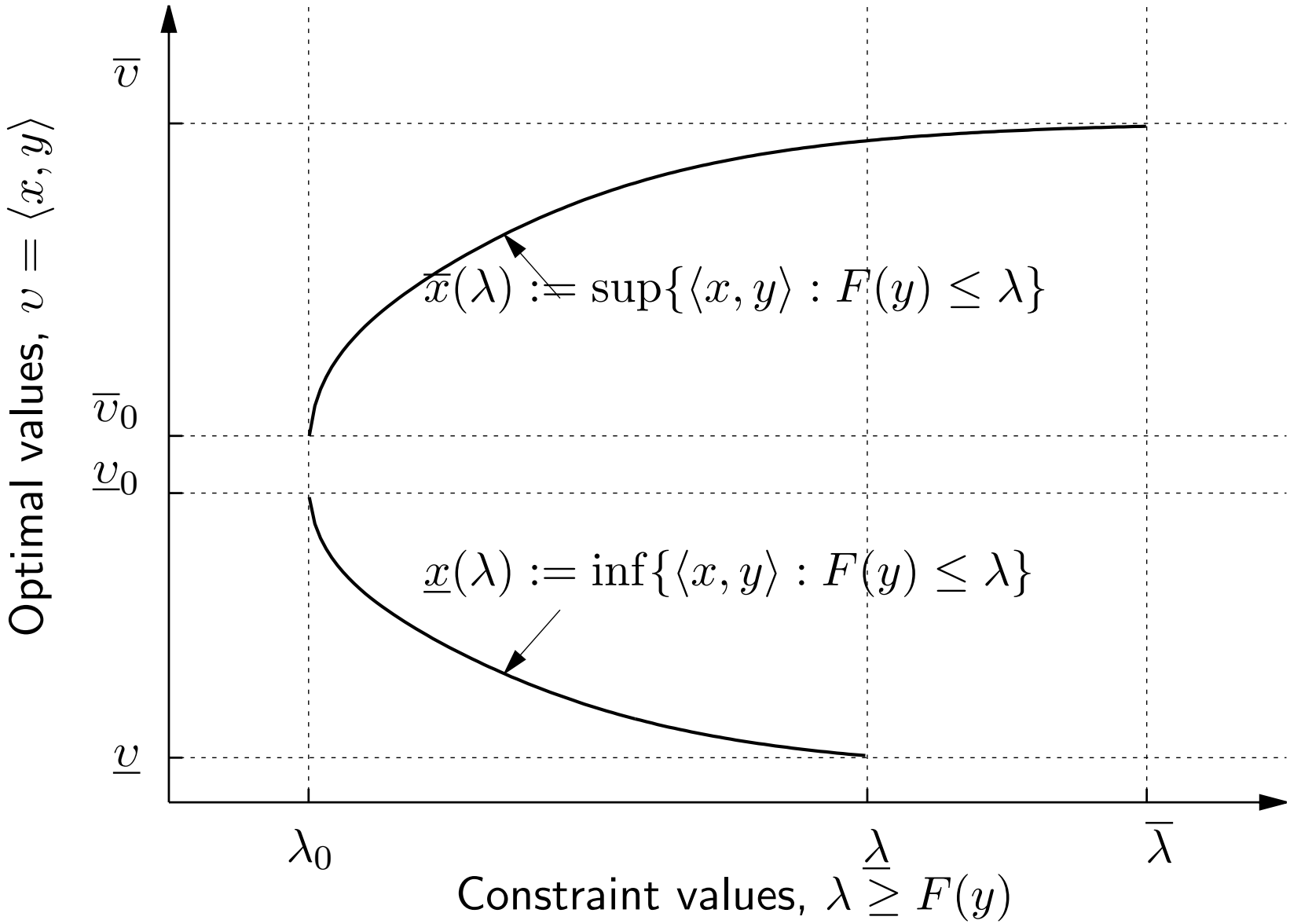
- Optimal solutions:

$$p(\beta) = e^{\beta\, u - \Psi(\beta)}\, q\,, \qquad \mathbb{E}_{p(\beta)}\{u\} = \upsilon \quad \Big(\mathbb{E}_p\{\ln(p/q) = \lambda\}\Big)$$

Concave and convex value functions



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- x — hidden, y — observed, z — control, $u(x,z)$ — utility:
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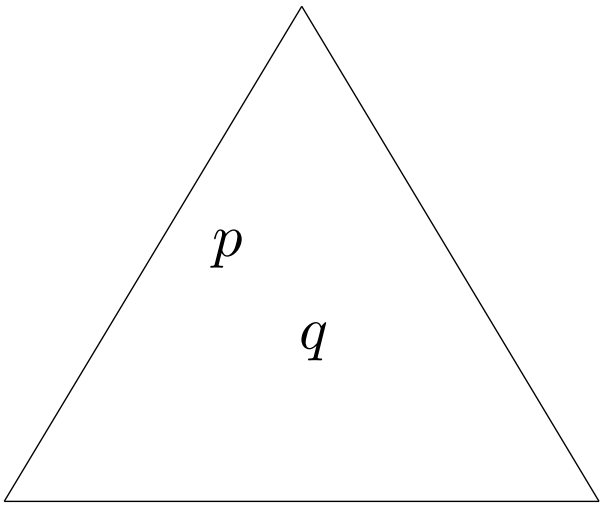
Value of Information (Stratonovich, 1965)

$$V(\lambda) := \overline{u}(\lambda) - \overline{u}(0) = \sup\{\mathbb{E}_w\{u\} : I\{x,z\} \leq \lambda\} - \text{const}$$

Variational problems for composite systems



$\mathcal{P}(X)$

type I : Find optimal $p \in \mathcal{P}(X)$

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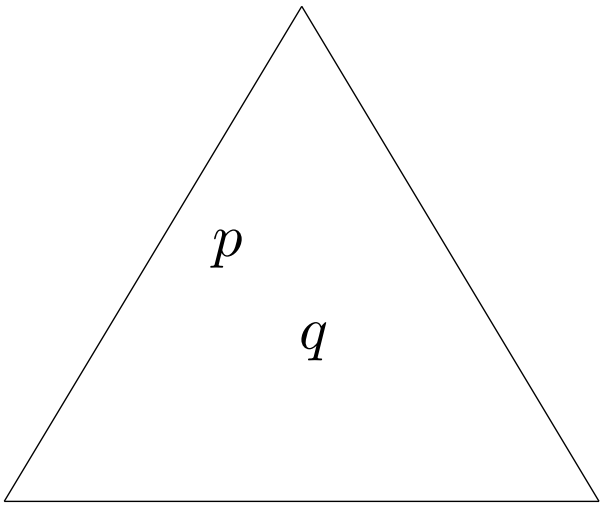

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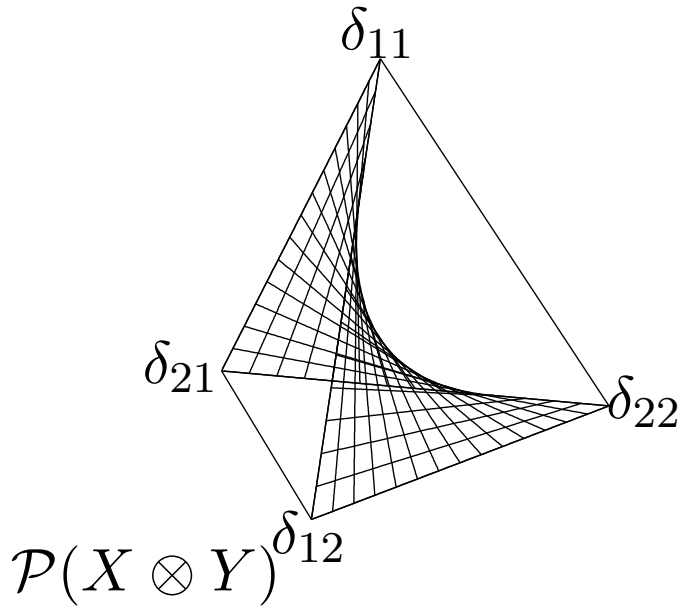

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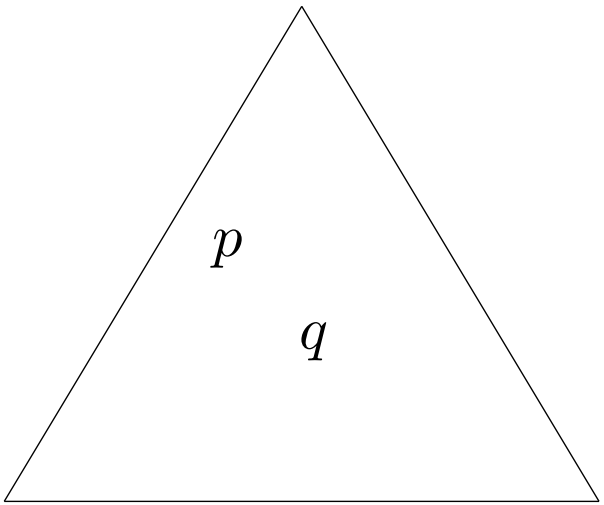

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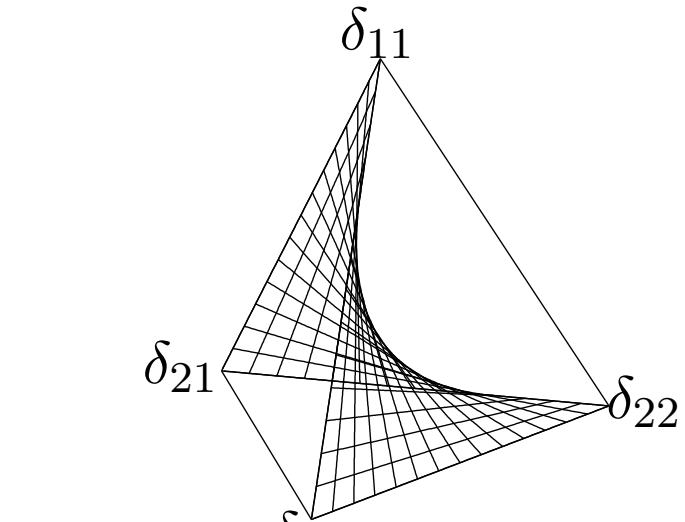

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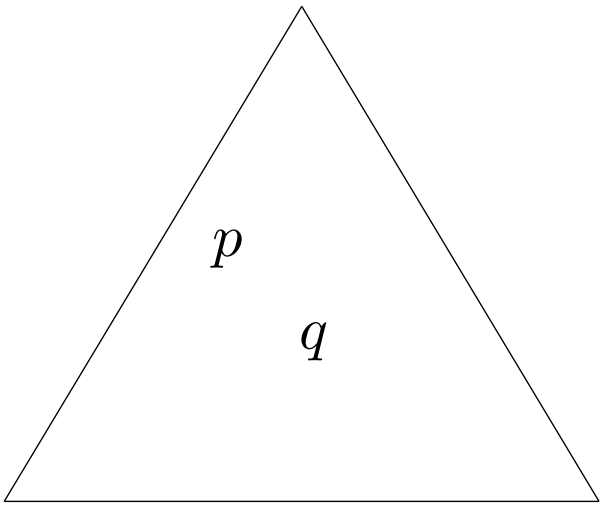

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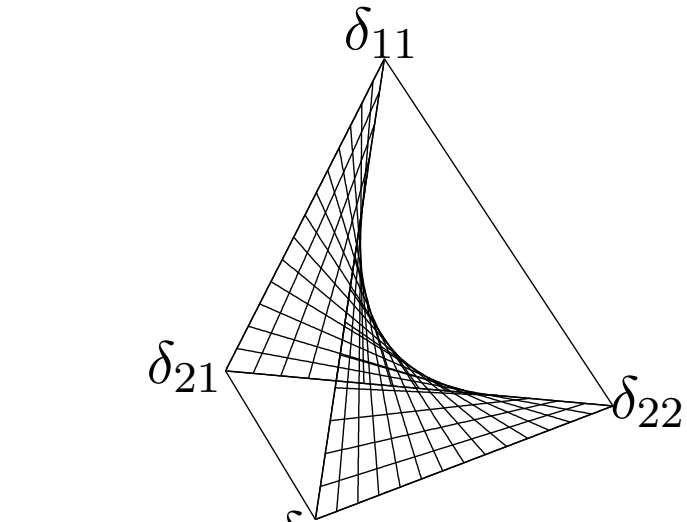

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Solving Problems by Iterative Procedures

Optimization and Search Problems

Given a set X (a search space) and an objective function $f : X \to \mathbb{R}$, find $\overline{x}$ such that $f(\overline{x}) = \upsilon$ (e.g. $f(\overline{x}) = \sup f(x)$ or $\nabla f(\overline{x}) = 0$, $\nabla^2 f(\overline{x}) \leq 0$).

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$$\Lambda(x_t) = x_t + \Delta x\,, \qquad \Delta x = -[\nabla^2 f(x_t)]^{-1} \nabla f(x_t) + \mathrm{rand}(X)$$

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Remark (Markov property)

Observe that $x_{t+1} = \Lambda(x_t)$ does not depend on $x_{t-1}, \ldots, x_1, x_0$.

Iterative procedures as Markov morphisms

Definition (Markov transition kernel from $(X, \mathcal{X})$ to $(Y, \mathcal{Y})$)

A conditional probability measure $p(Y_i \mid x)$ on $(Y, \mathcal{Y})$ that is $\mathcal{X}$-measurable for each $Y_i \in \mathcal{Y}$.

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Randomized (probabilistic) transitions are such that $p(x_{t+1} \mid x_t) > 0$ for several x_{t+1} and some x_t.

Optimal learning trajectory

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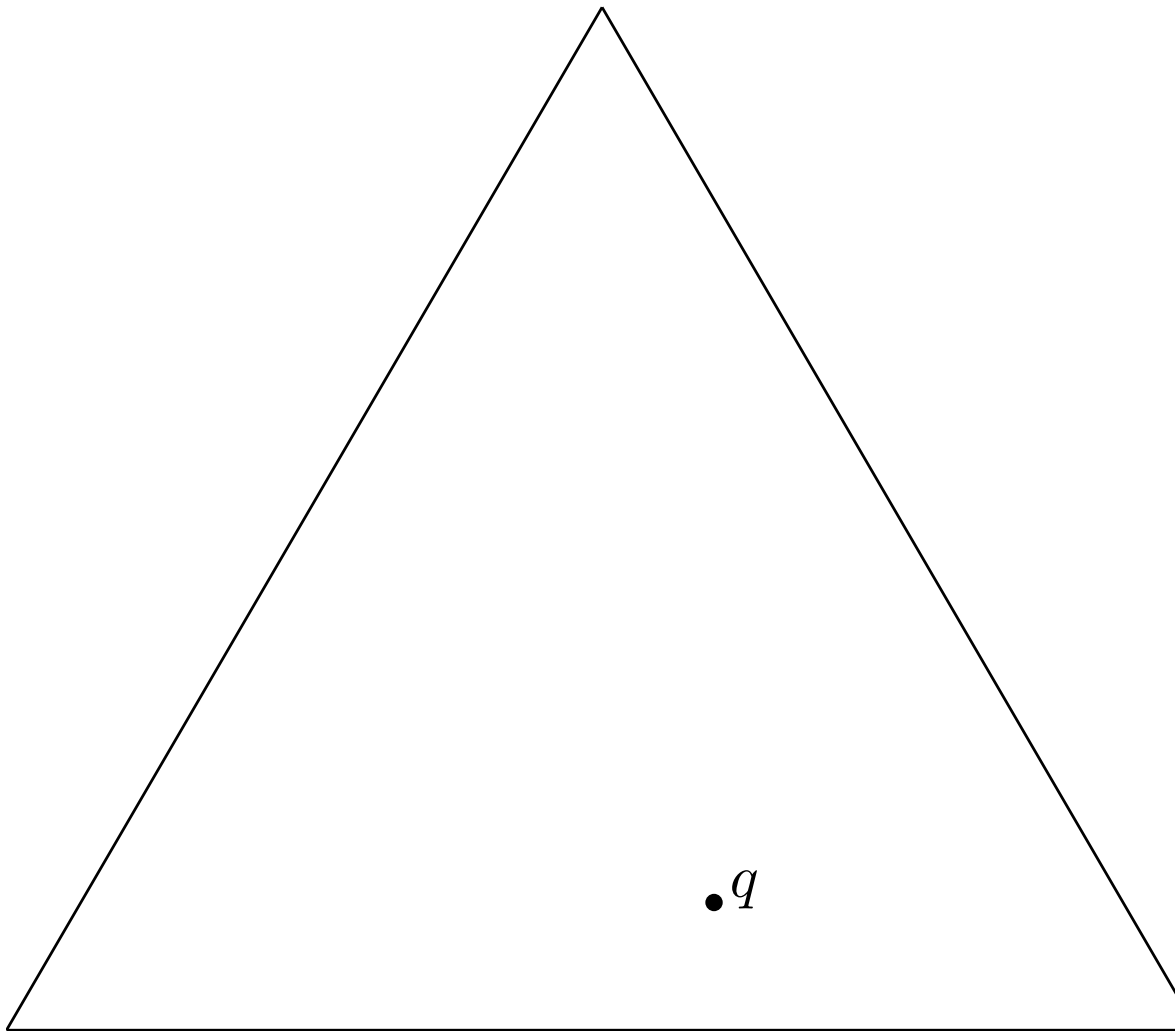
- Evolution
  $\Lambda : \mathcal{P}(X) \to \mathcal{P}(X)$:

$$q(t+1) = \Lambda\, q(t)$$



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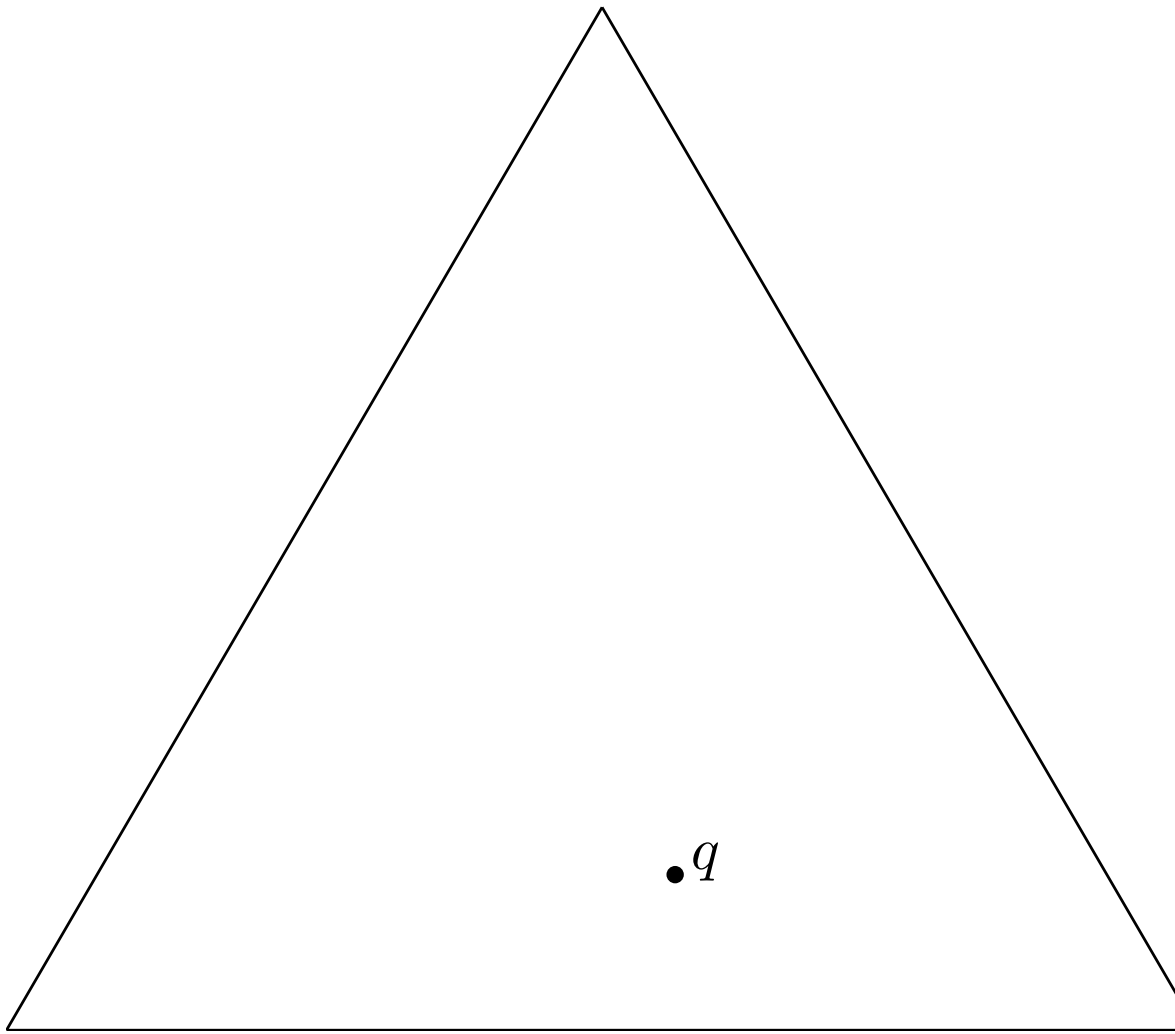

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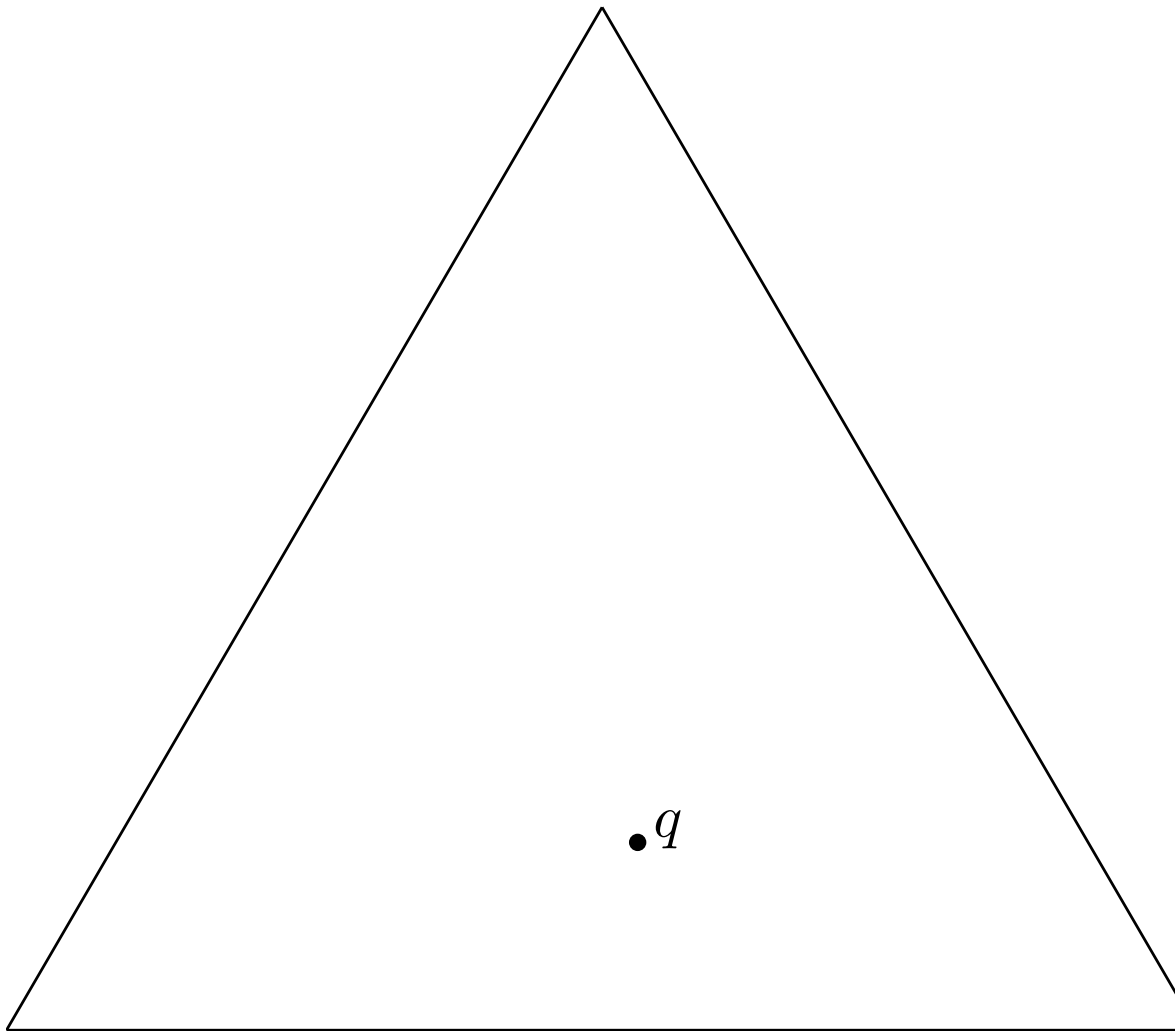

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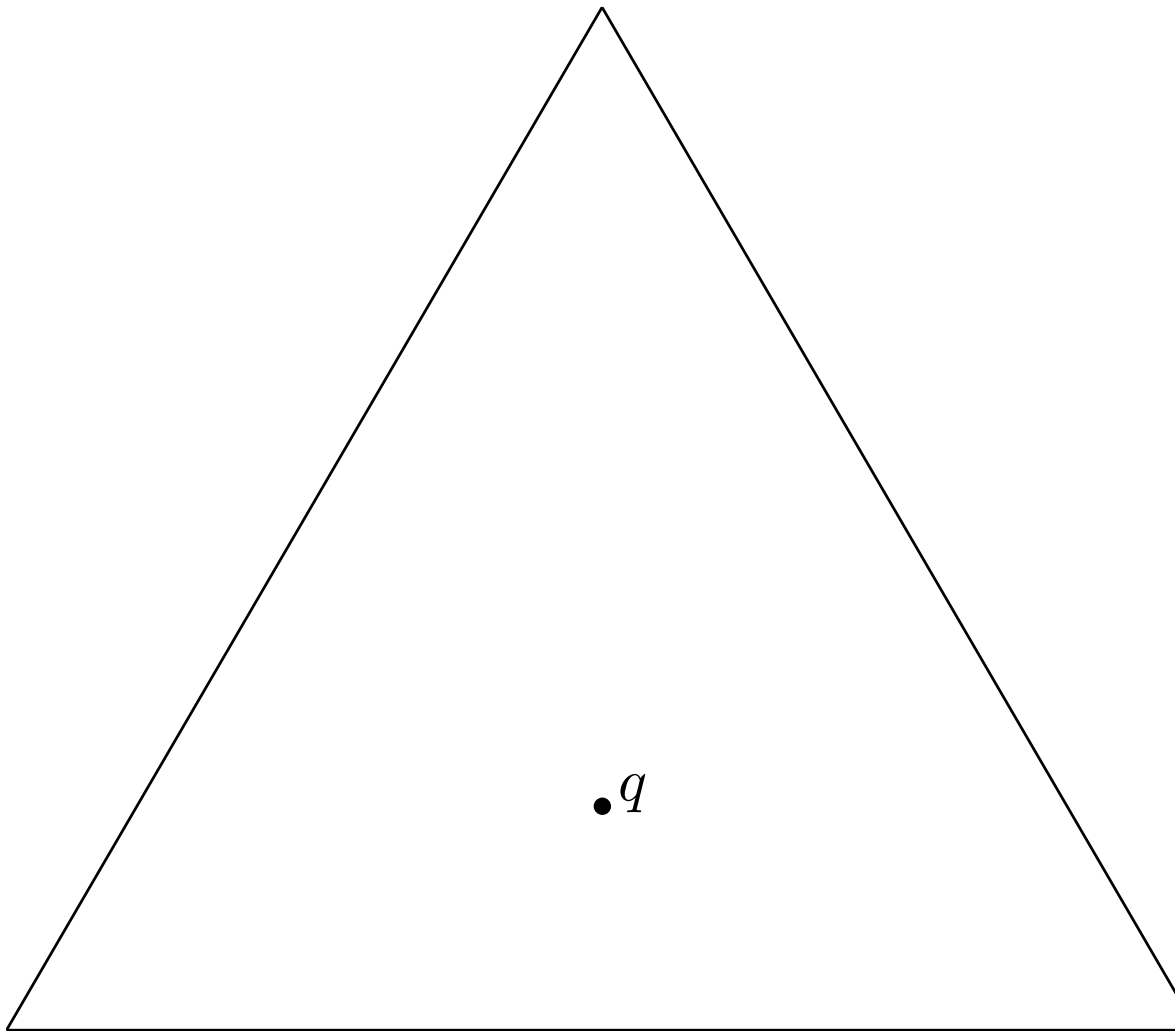

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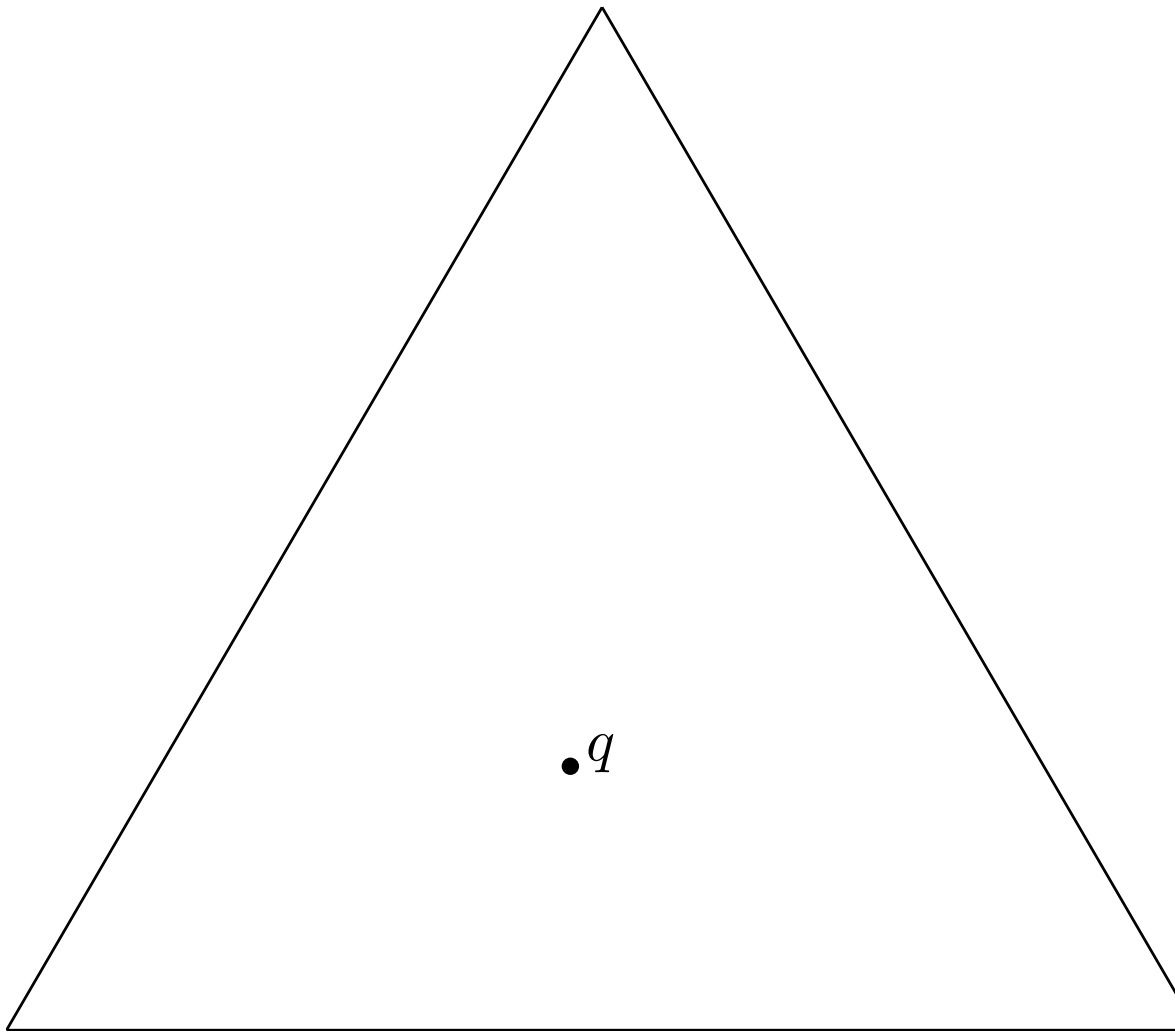

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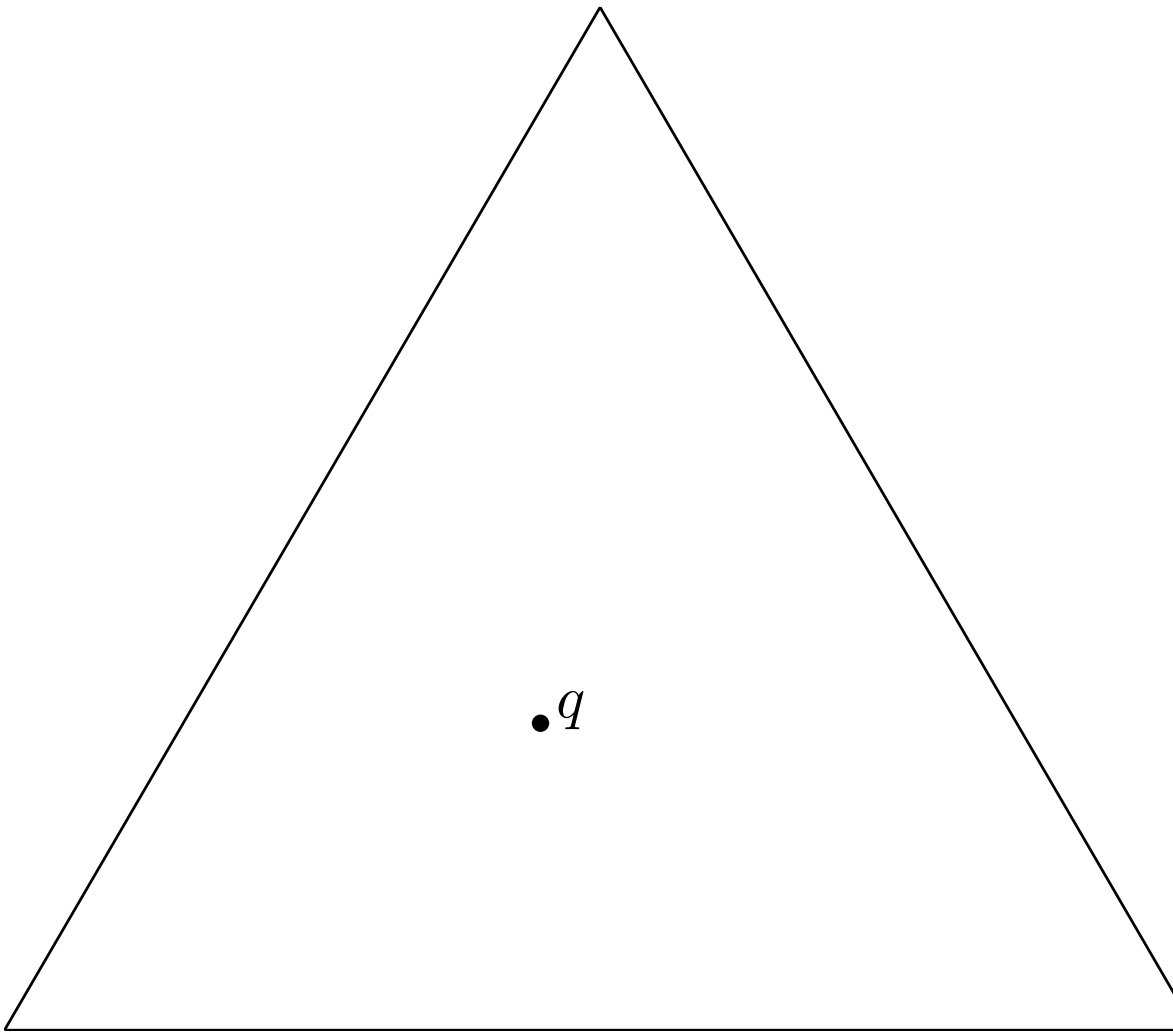

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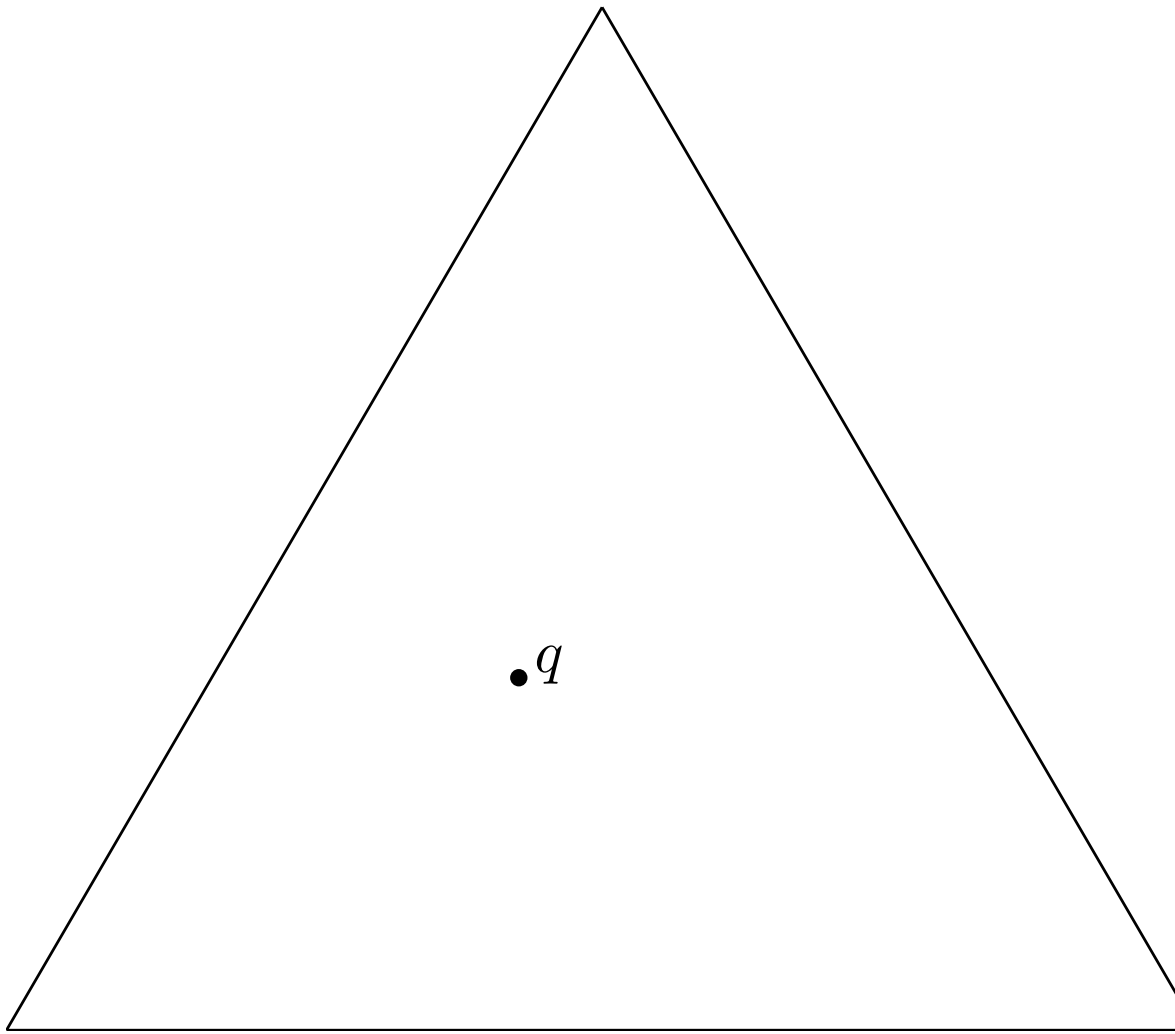

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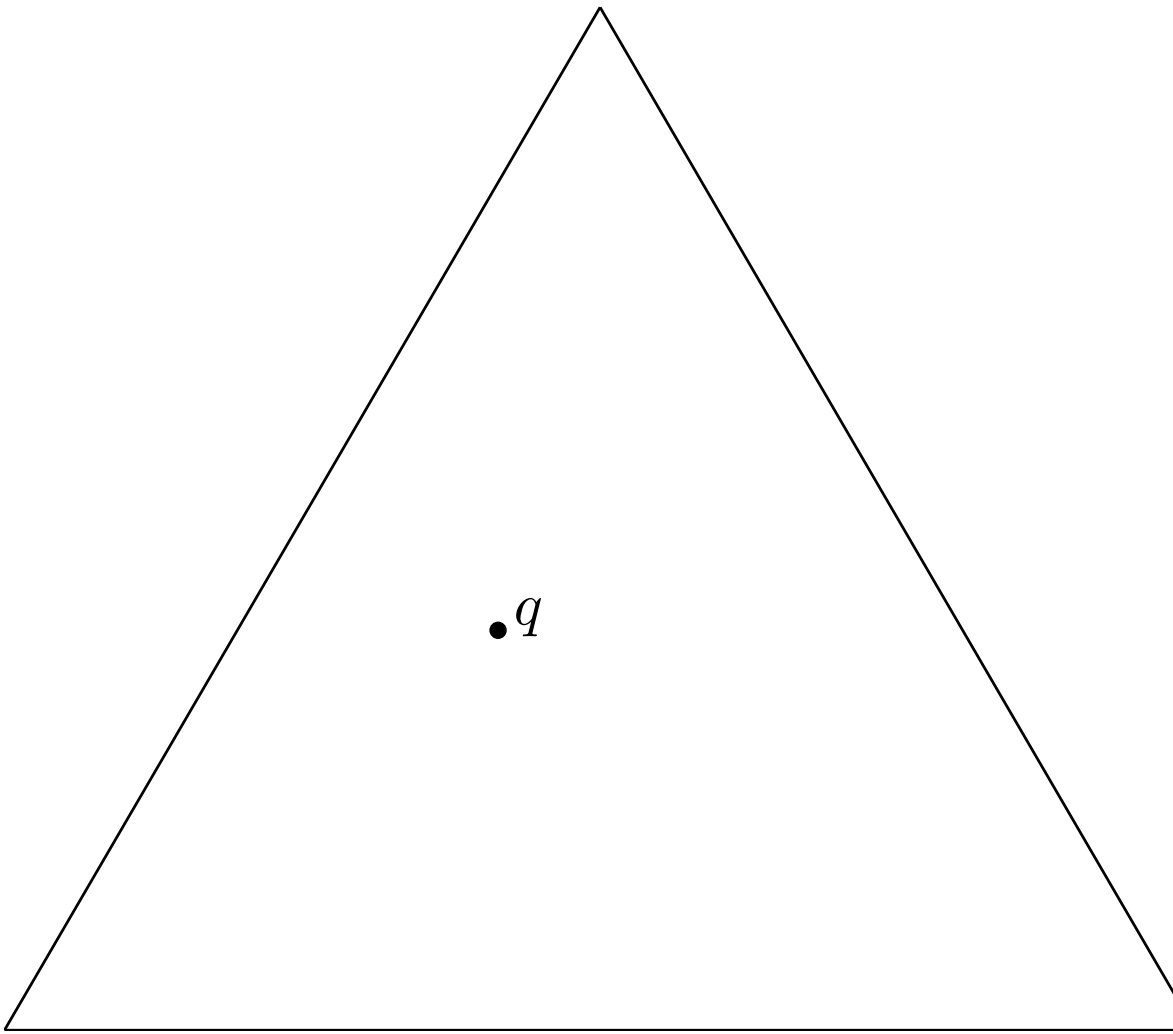

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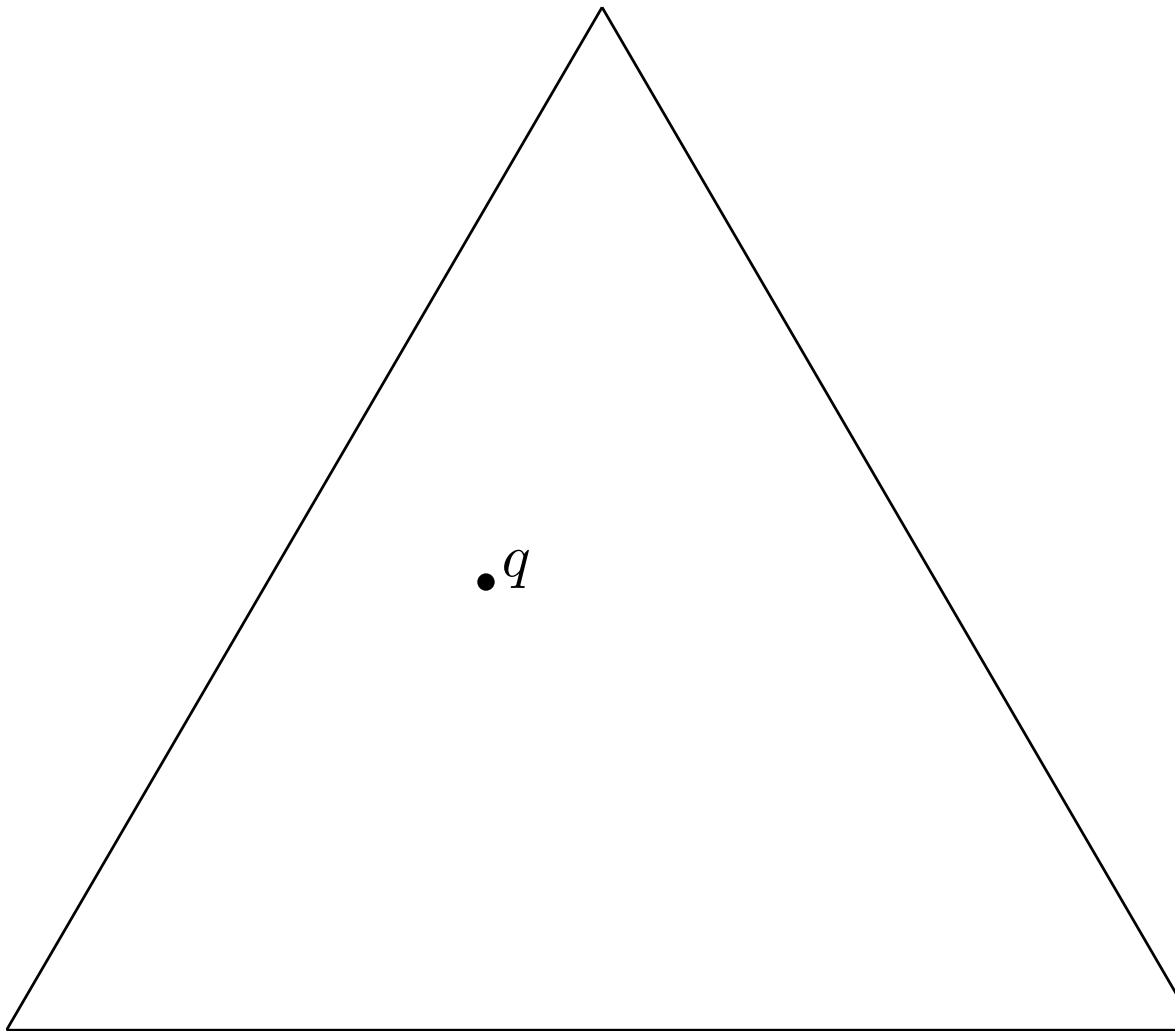

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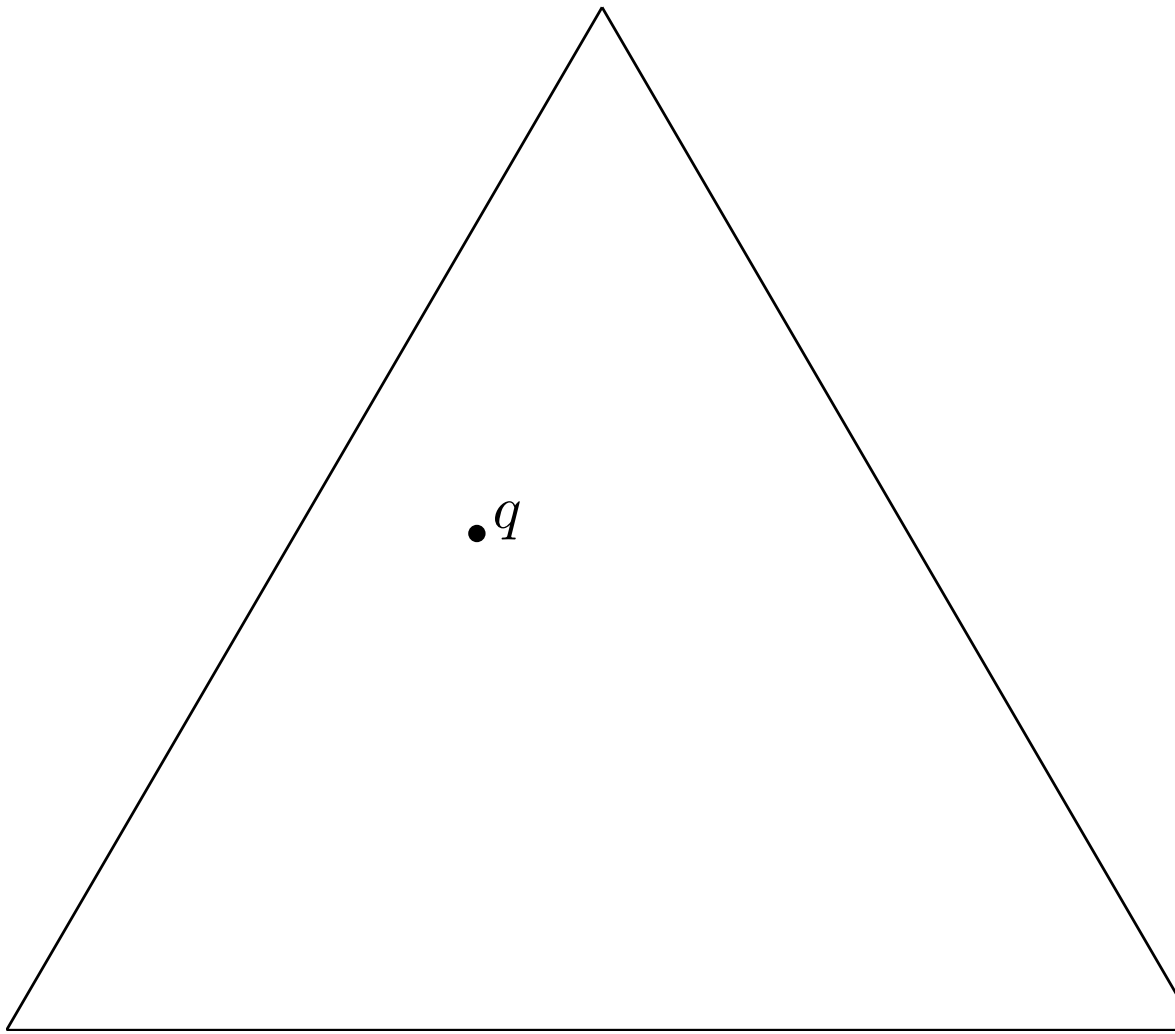

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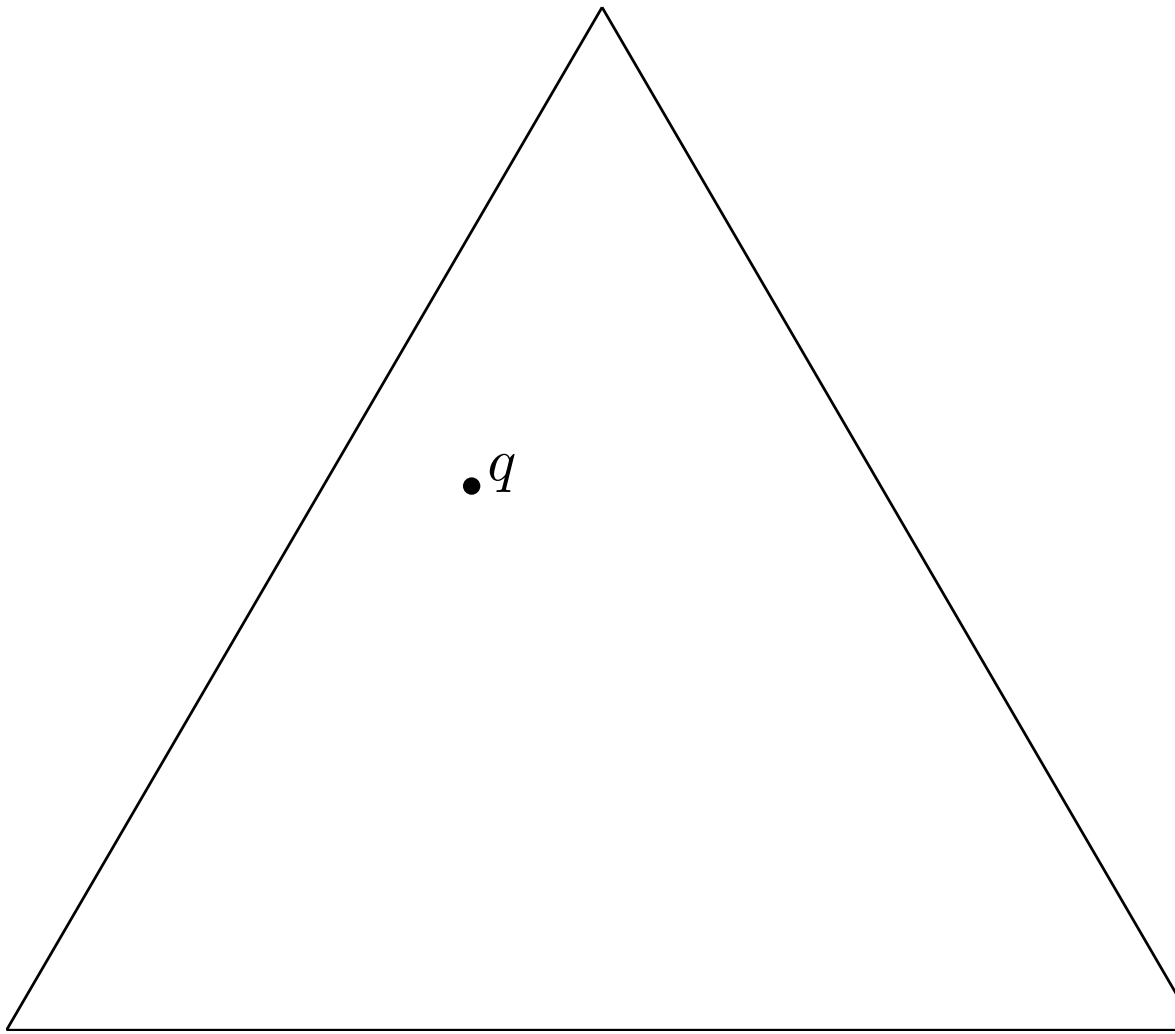

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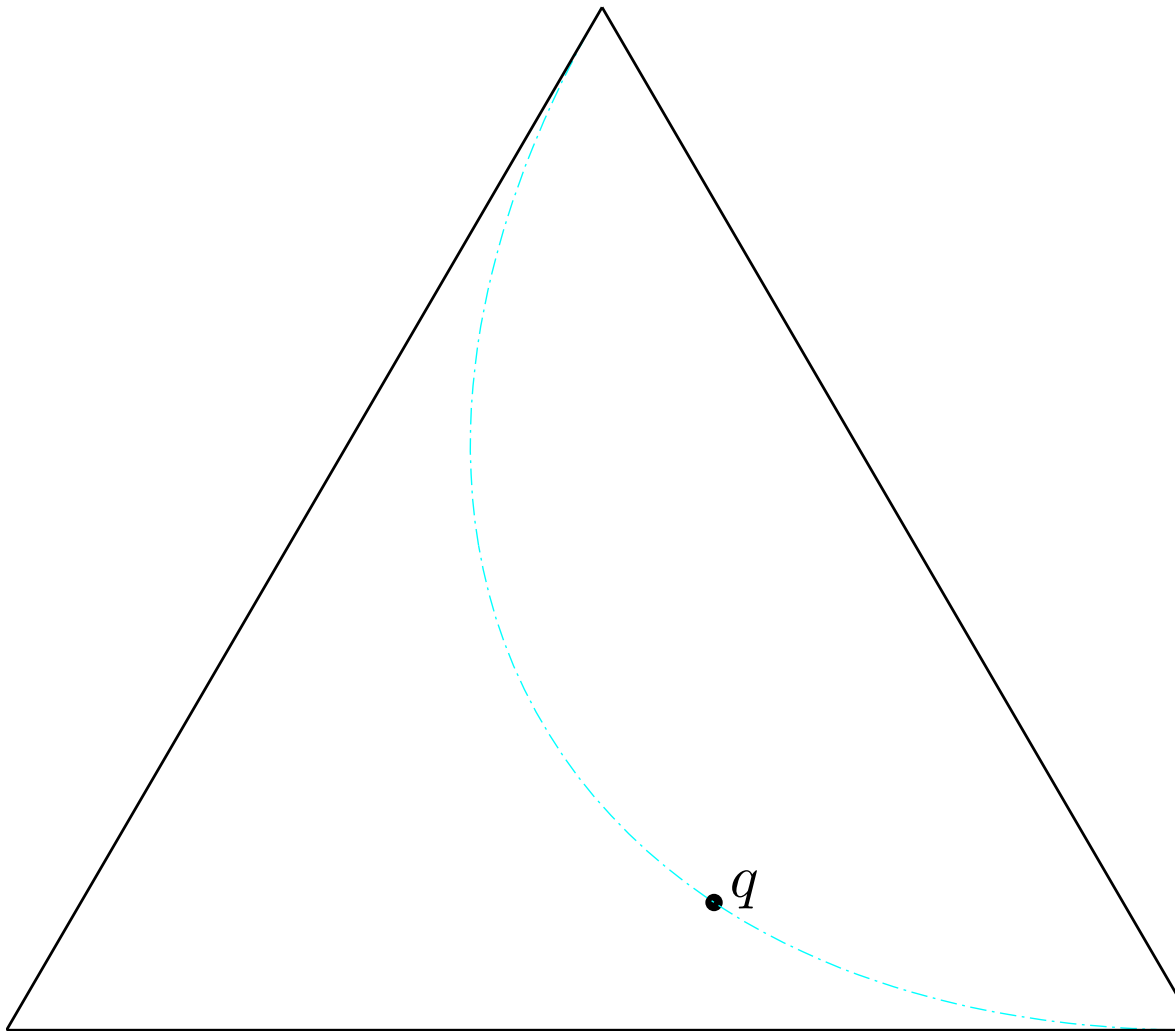

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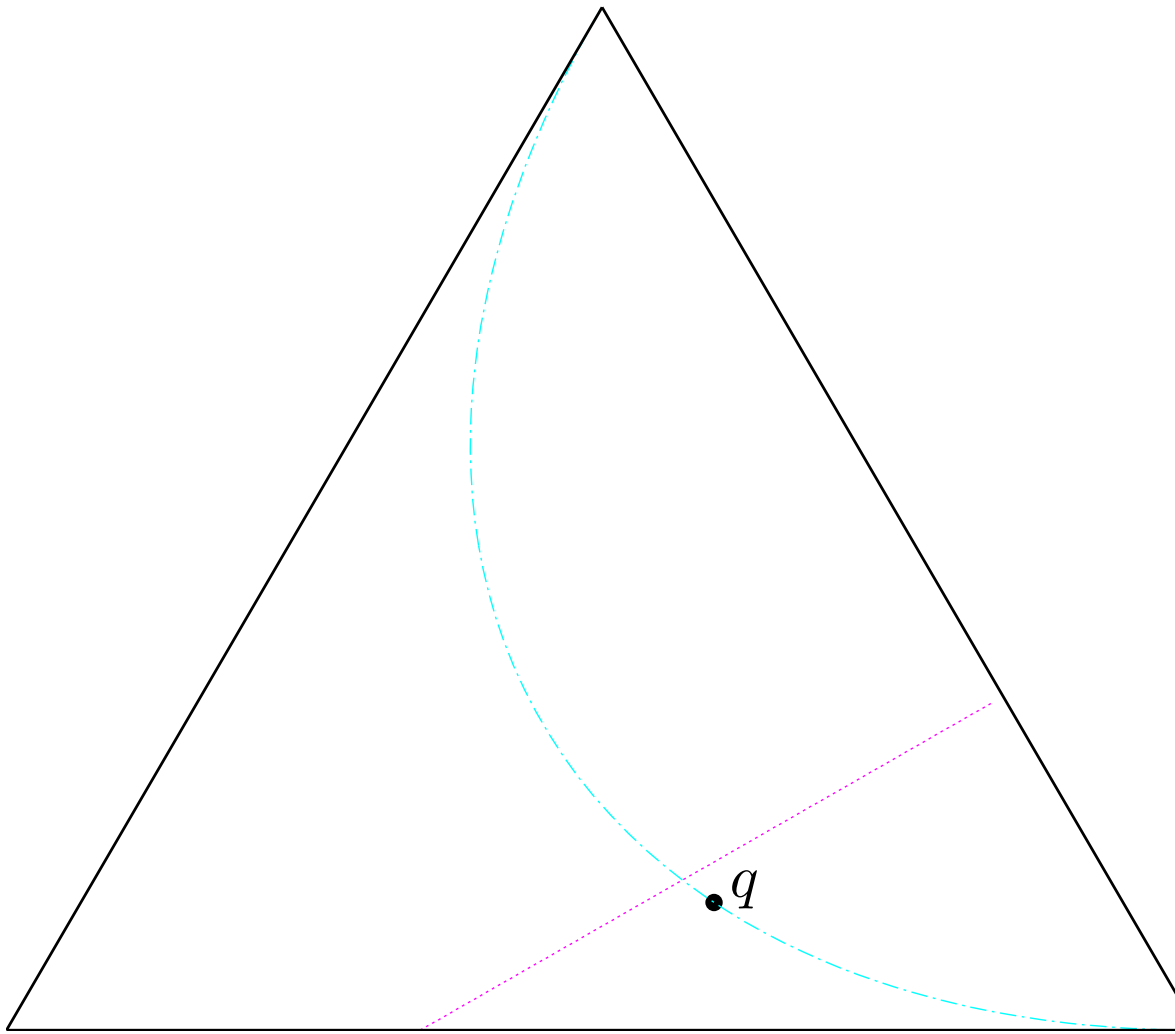

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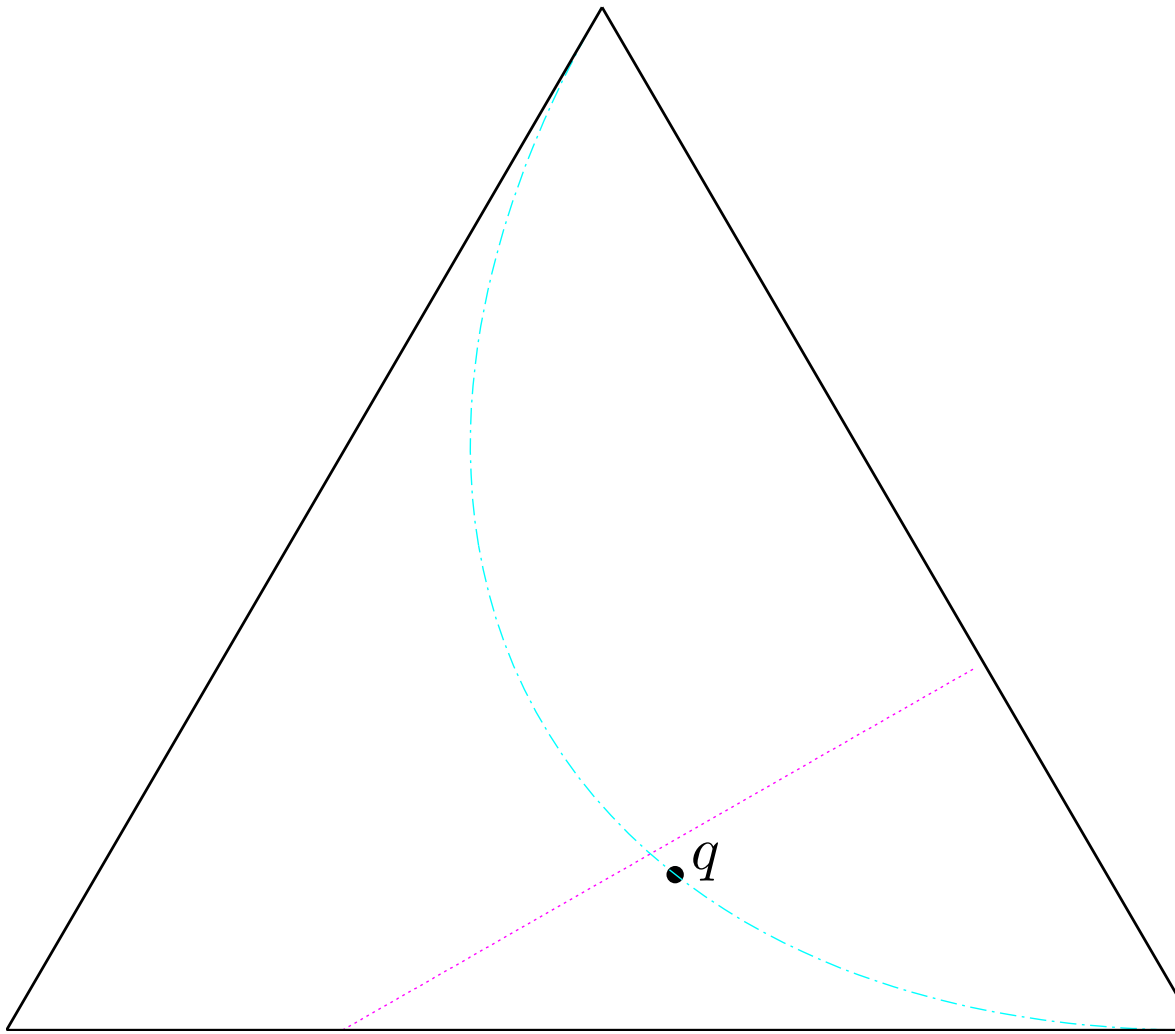

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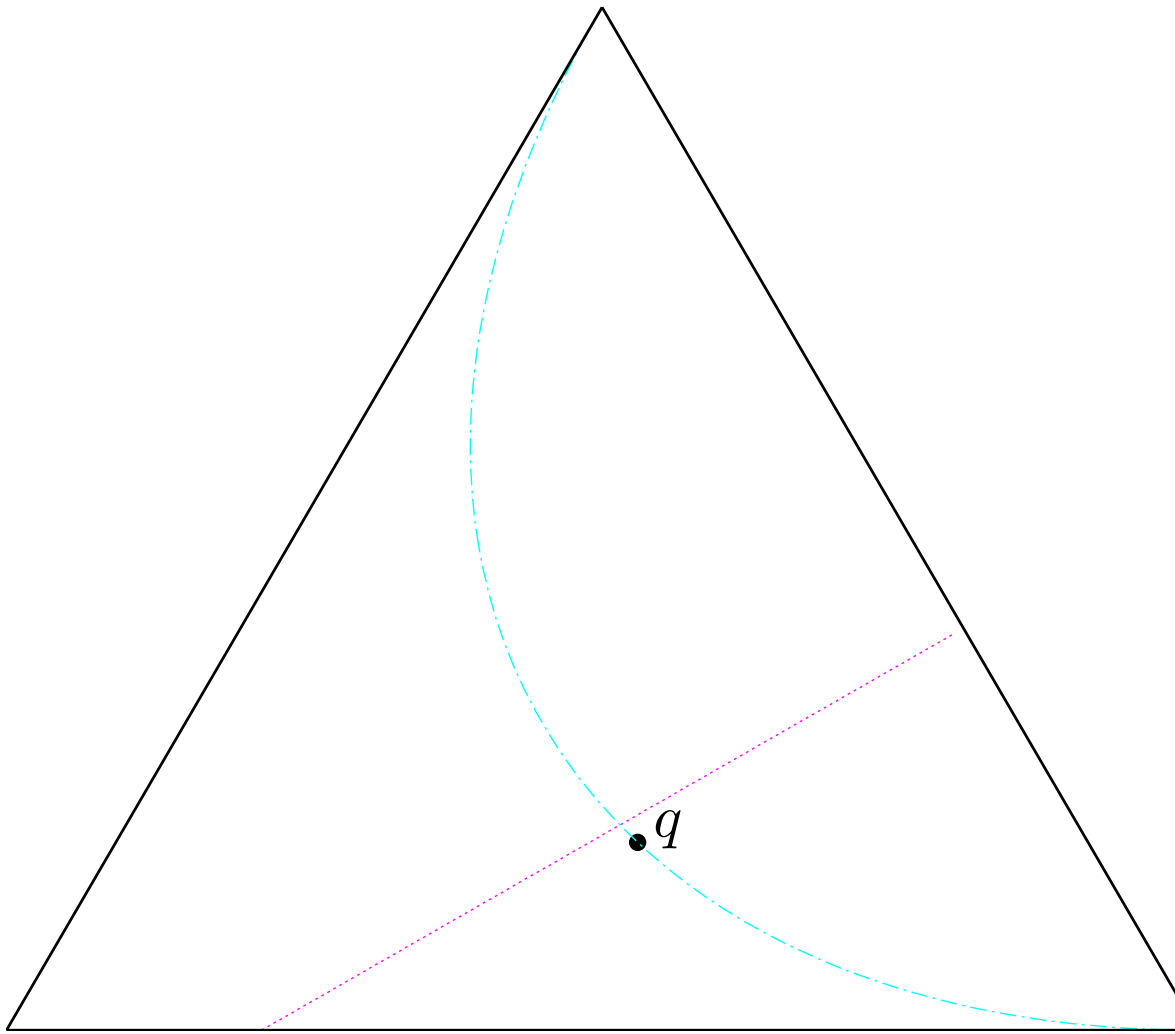

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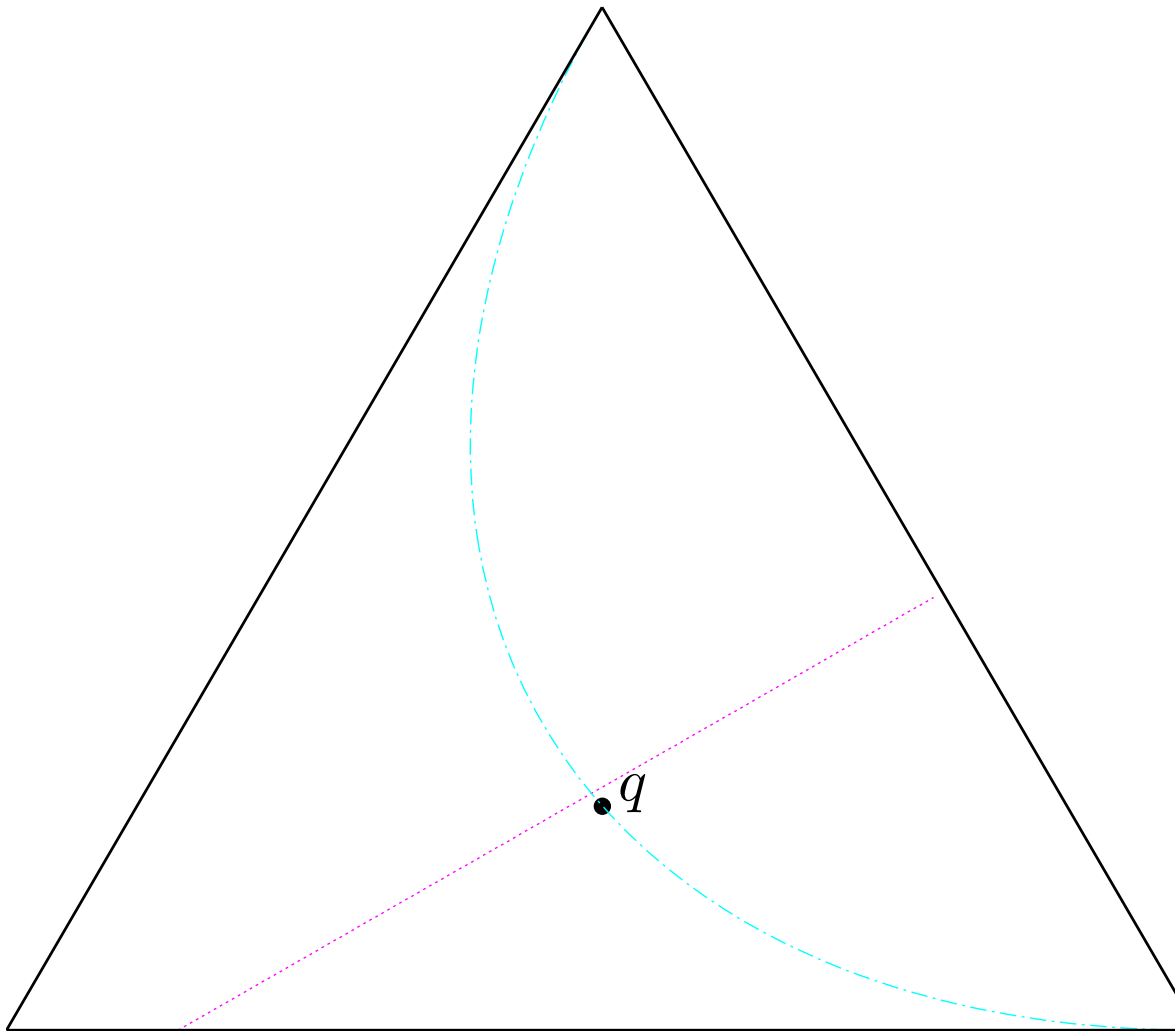

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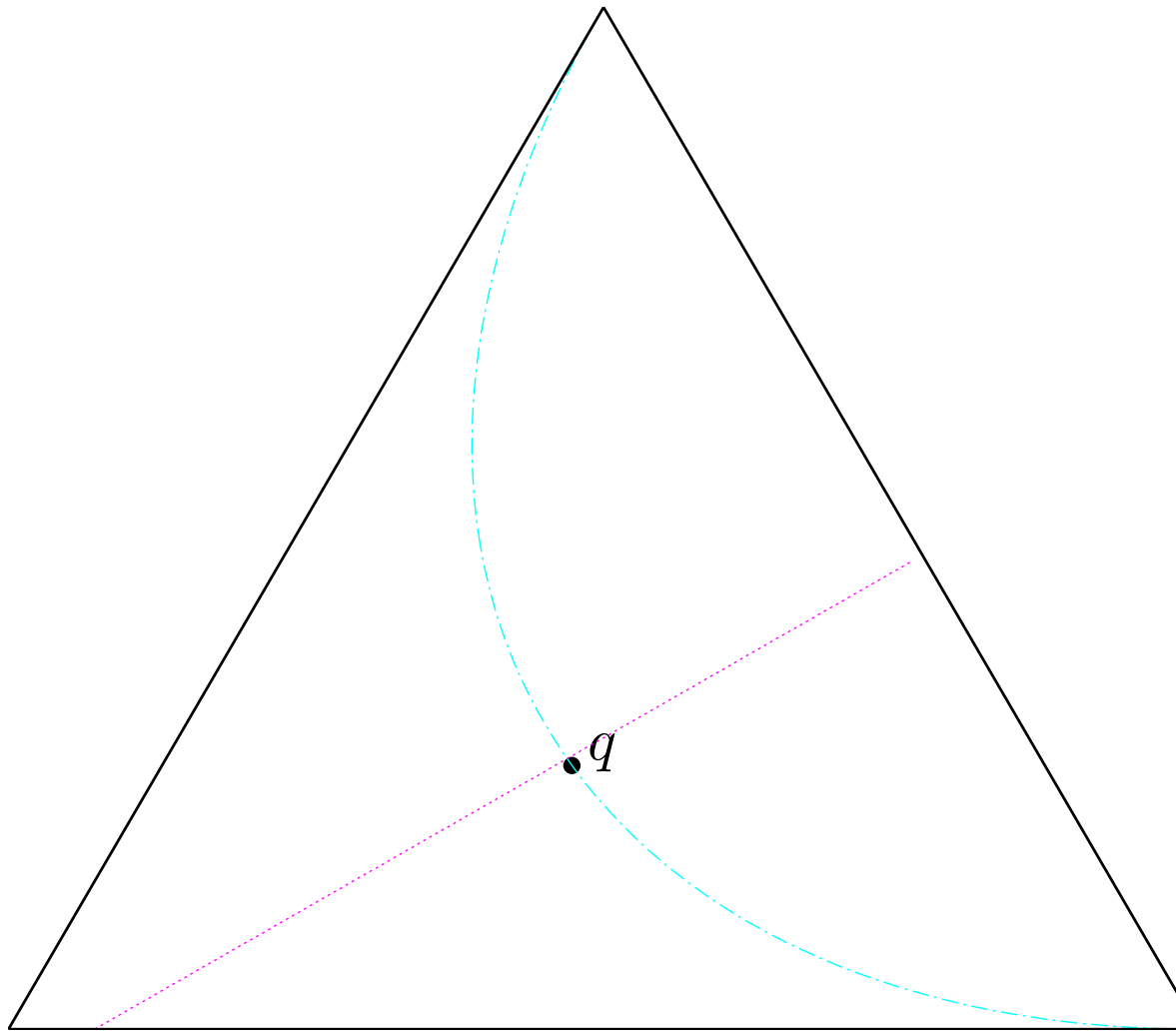

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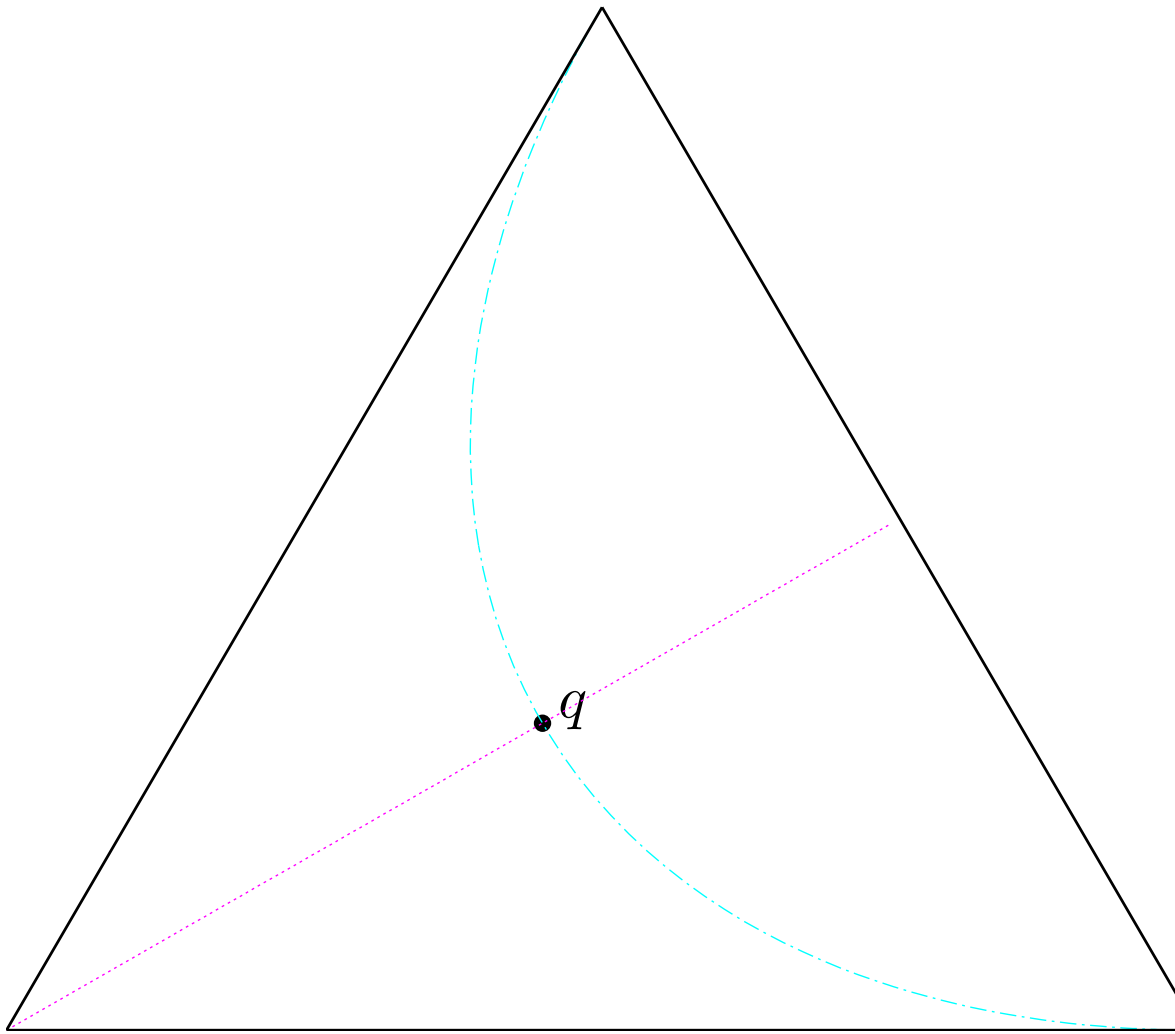

Optimal learning trajectory

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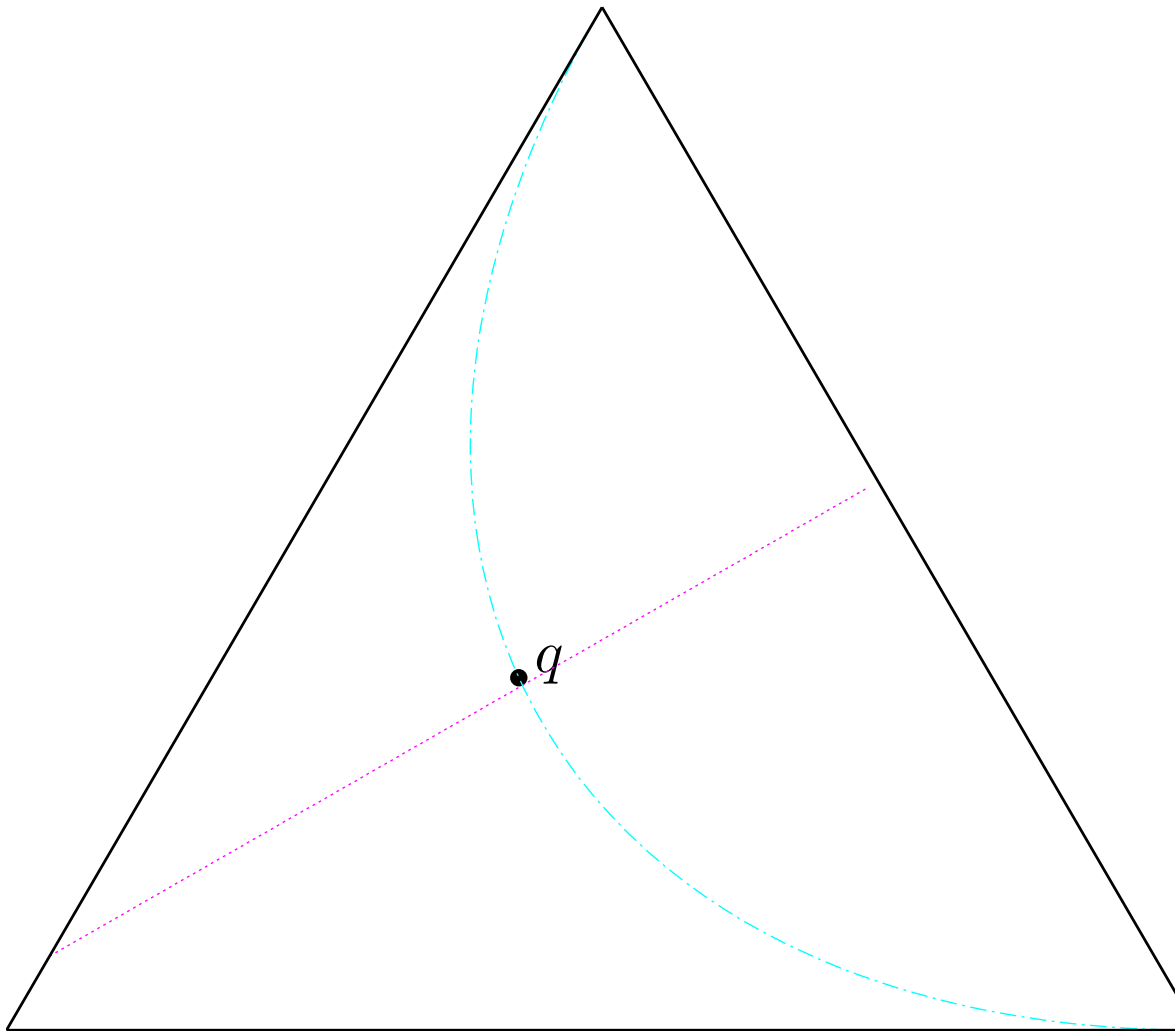

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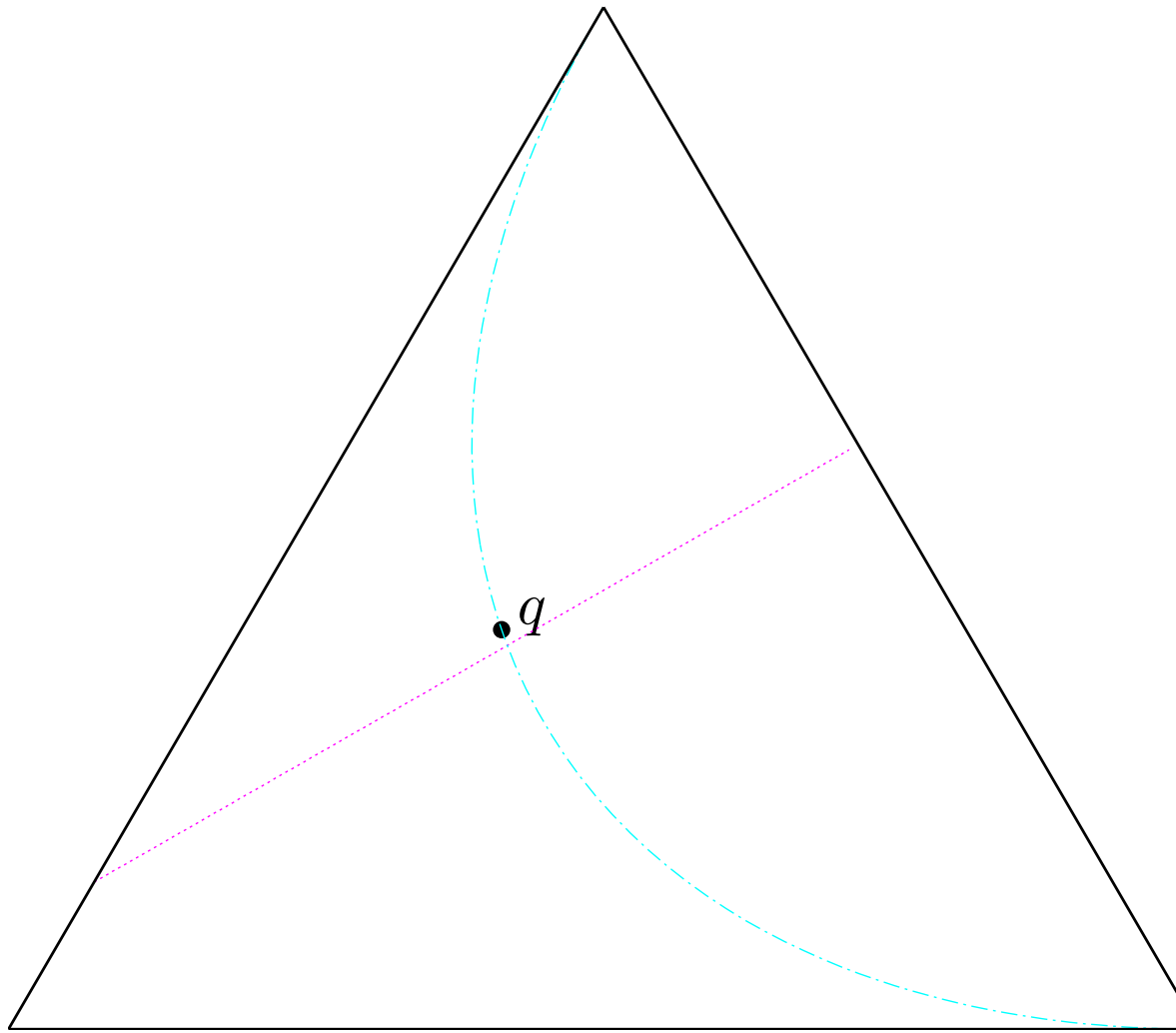

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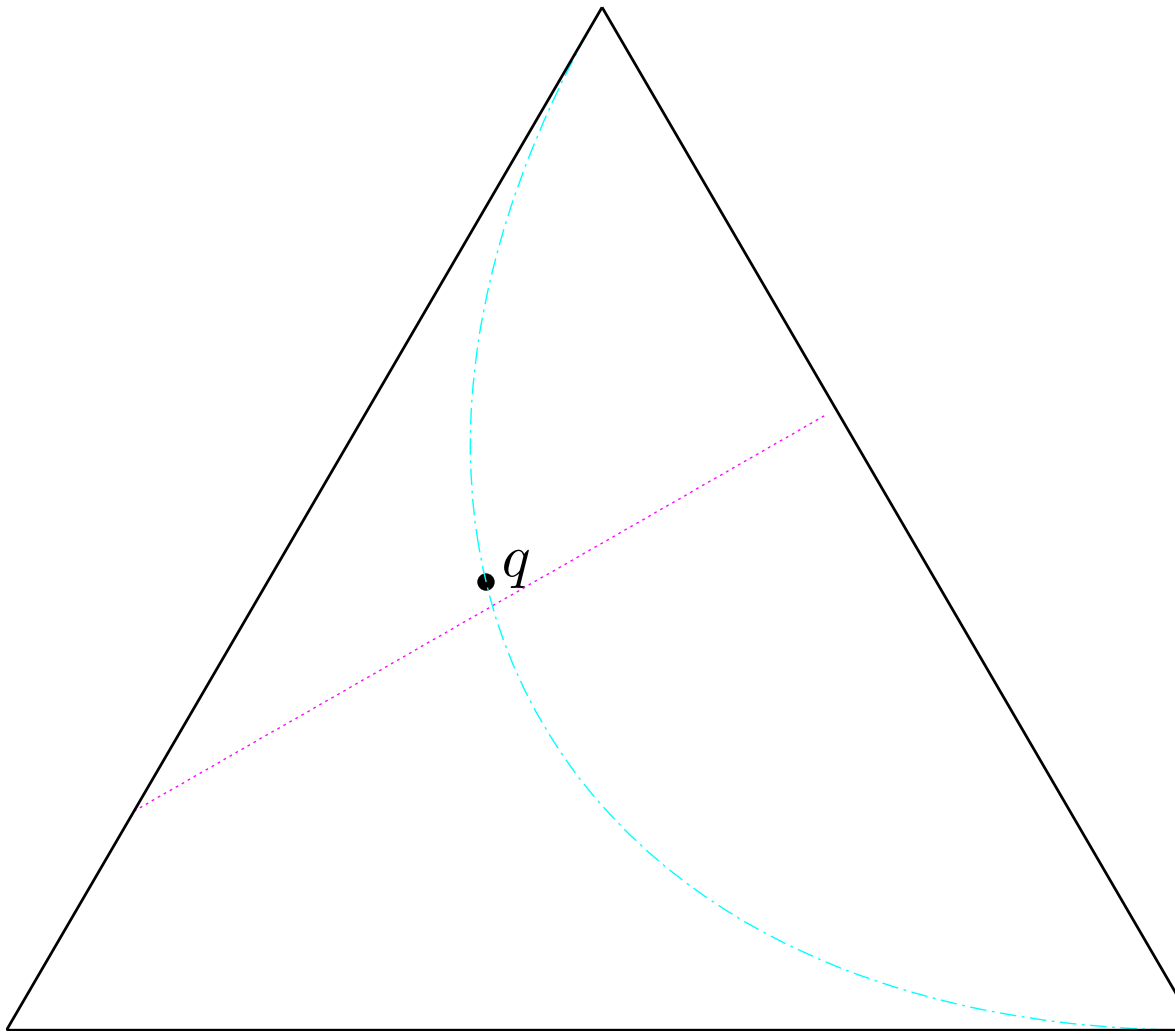

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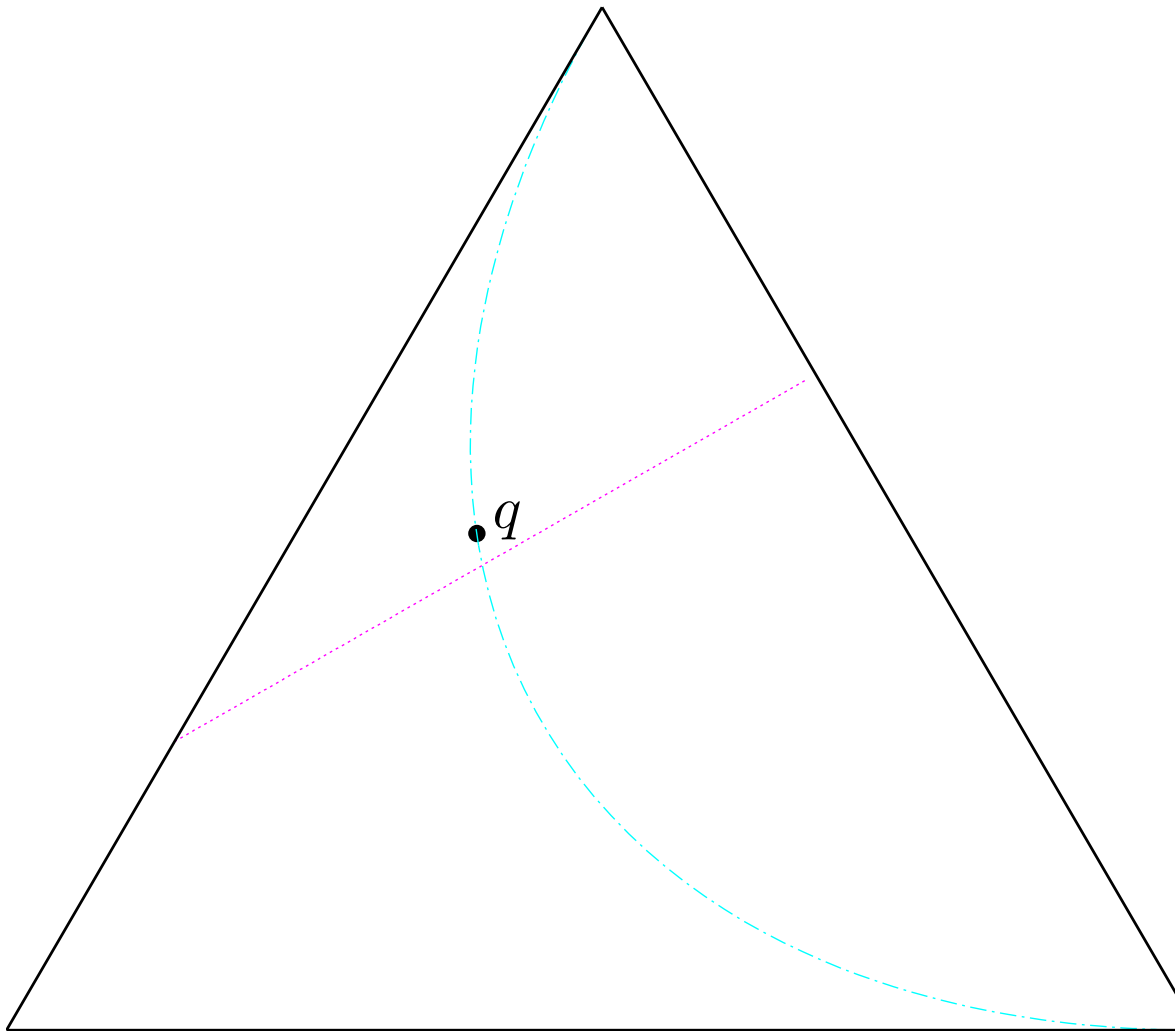

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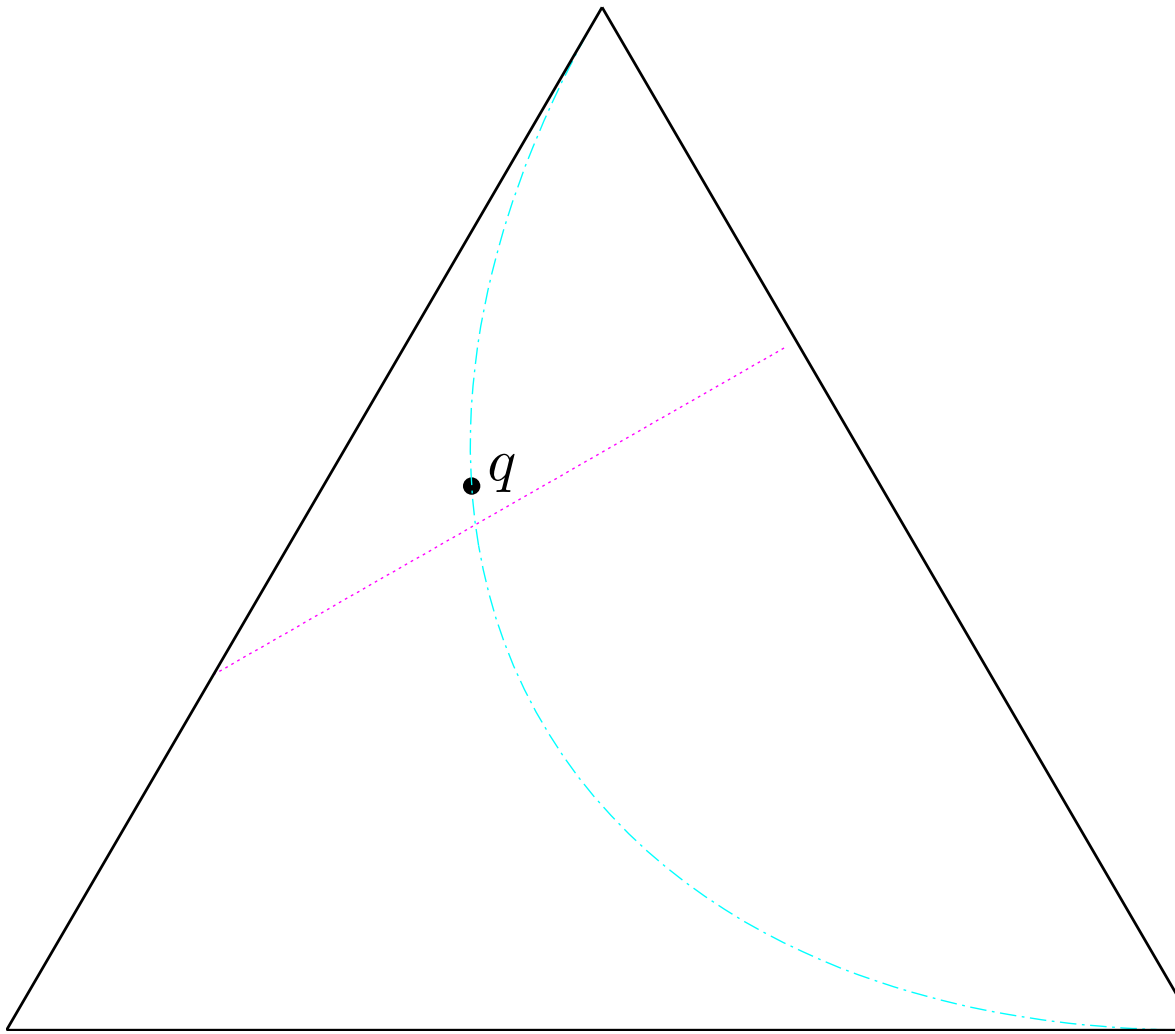

Optimal learning trajectory

Third variational problem

- Linear programming problem:

$$\text{maximize} \quad \mathbb{E}_w\{u(x,z)\} \quad \text{subject to} \quad I\{x,z\} \leq \lambda$$

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- Information:

$$I\{x,z\} = H\{z\} - H\{z \mid x\} \leq H\{z\} \leq \ln|Z|$$

Optimal transition kernels

- Optimal solutions achieving $V(\lambda)$ have exponential form, such as:

$$P(z \mid x) = \frac{e^{\beta u(x,z)}}{\sum_z e^{\beta u(x,z)}}$$

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- One can show that temperature β^{-1} is the slope of $V(\lambda)$:

$$\beta^{-1} = \frac{dV(\lambda)}{d\lambda}$$

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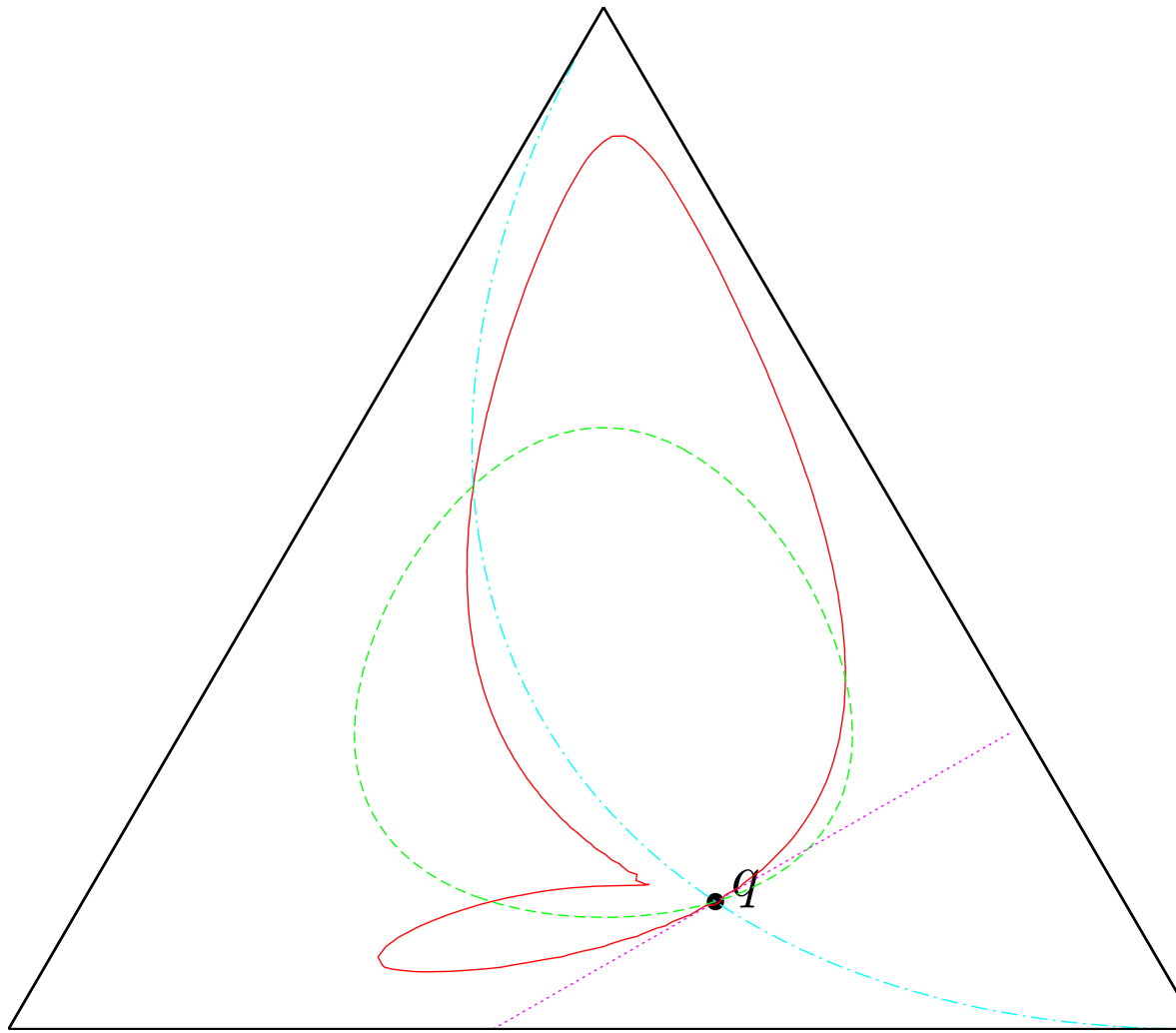

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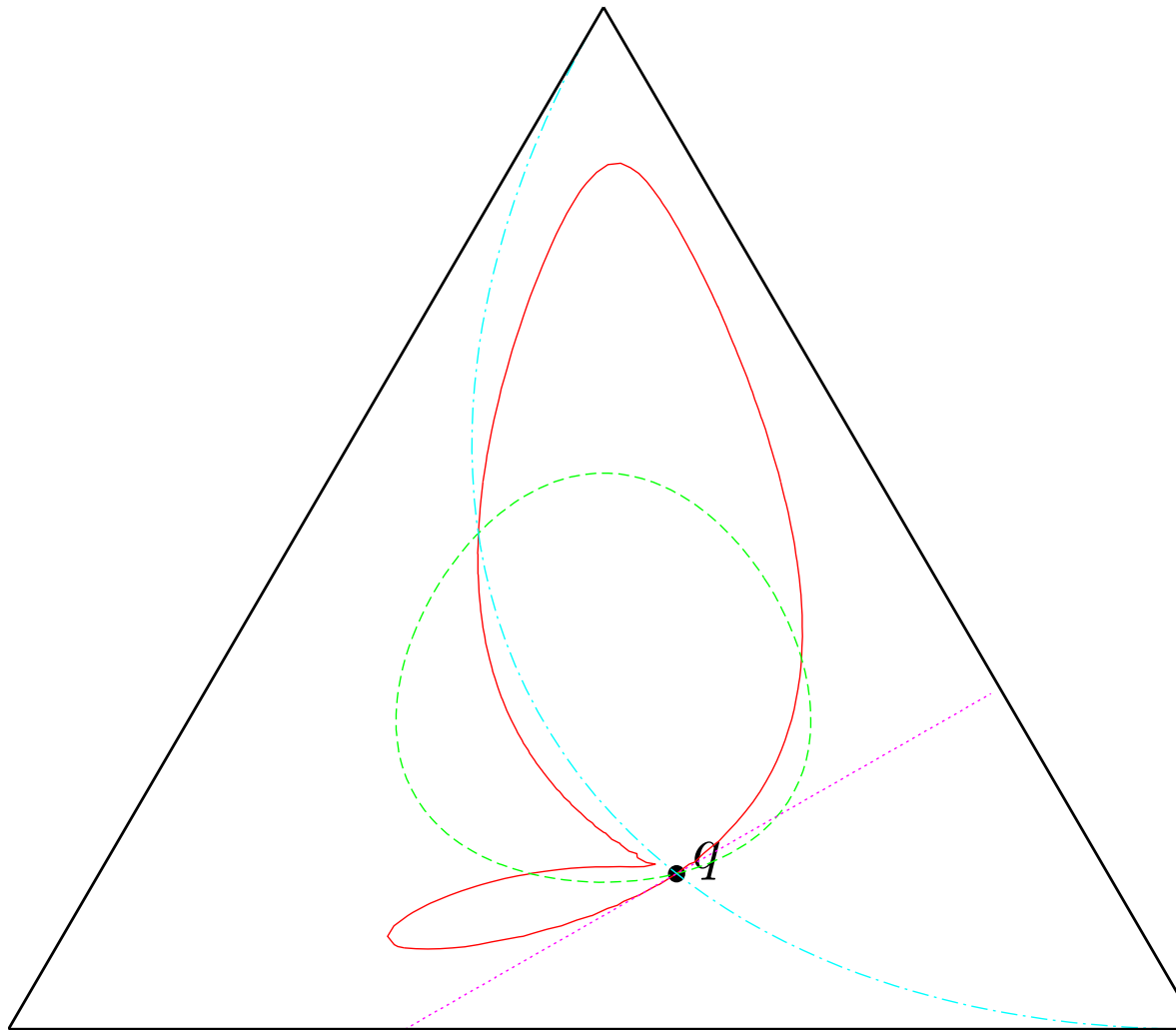

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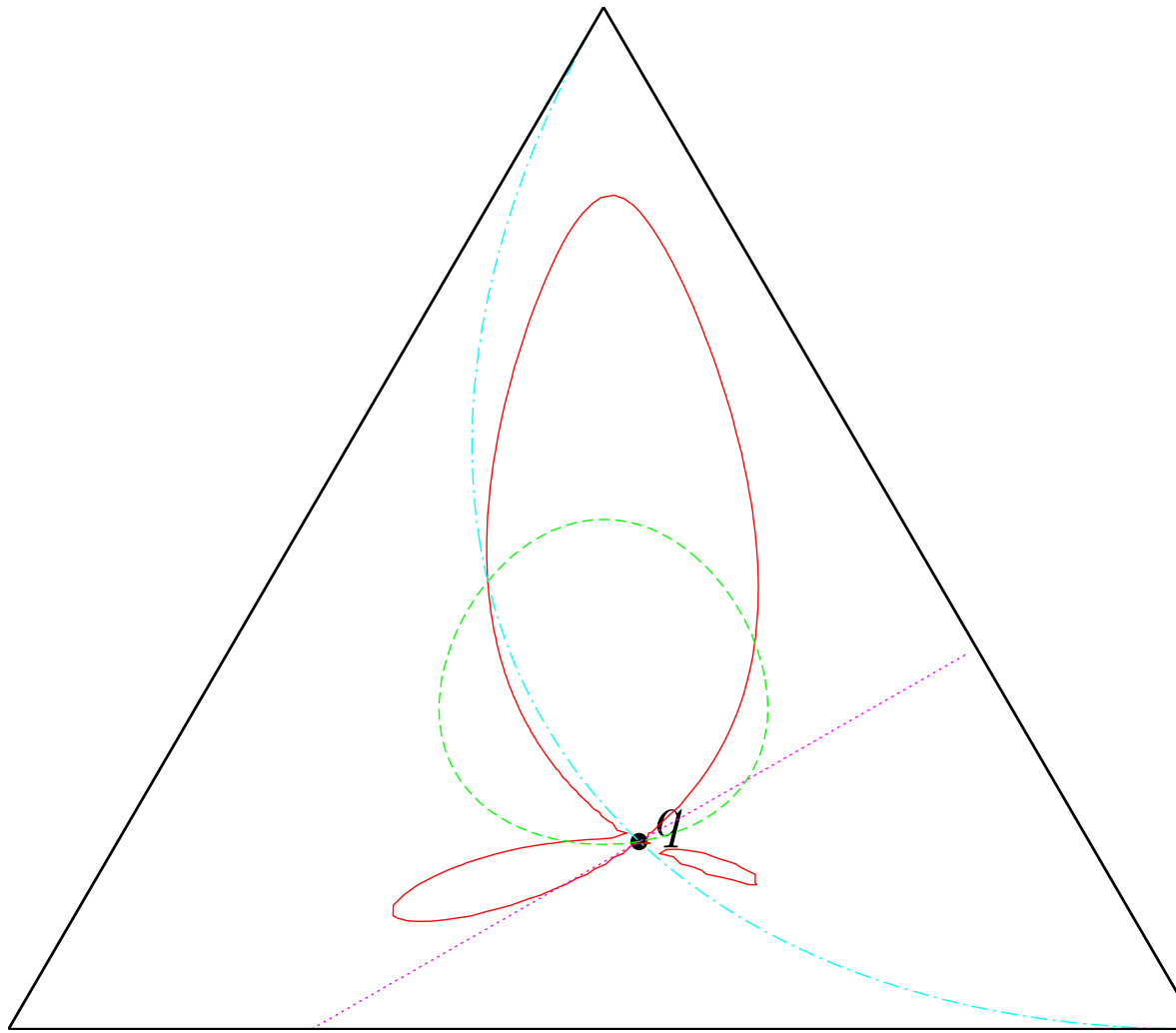

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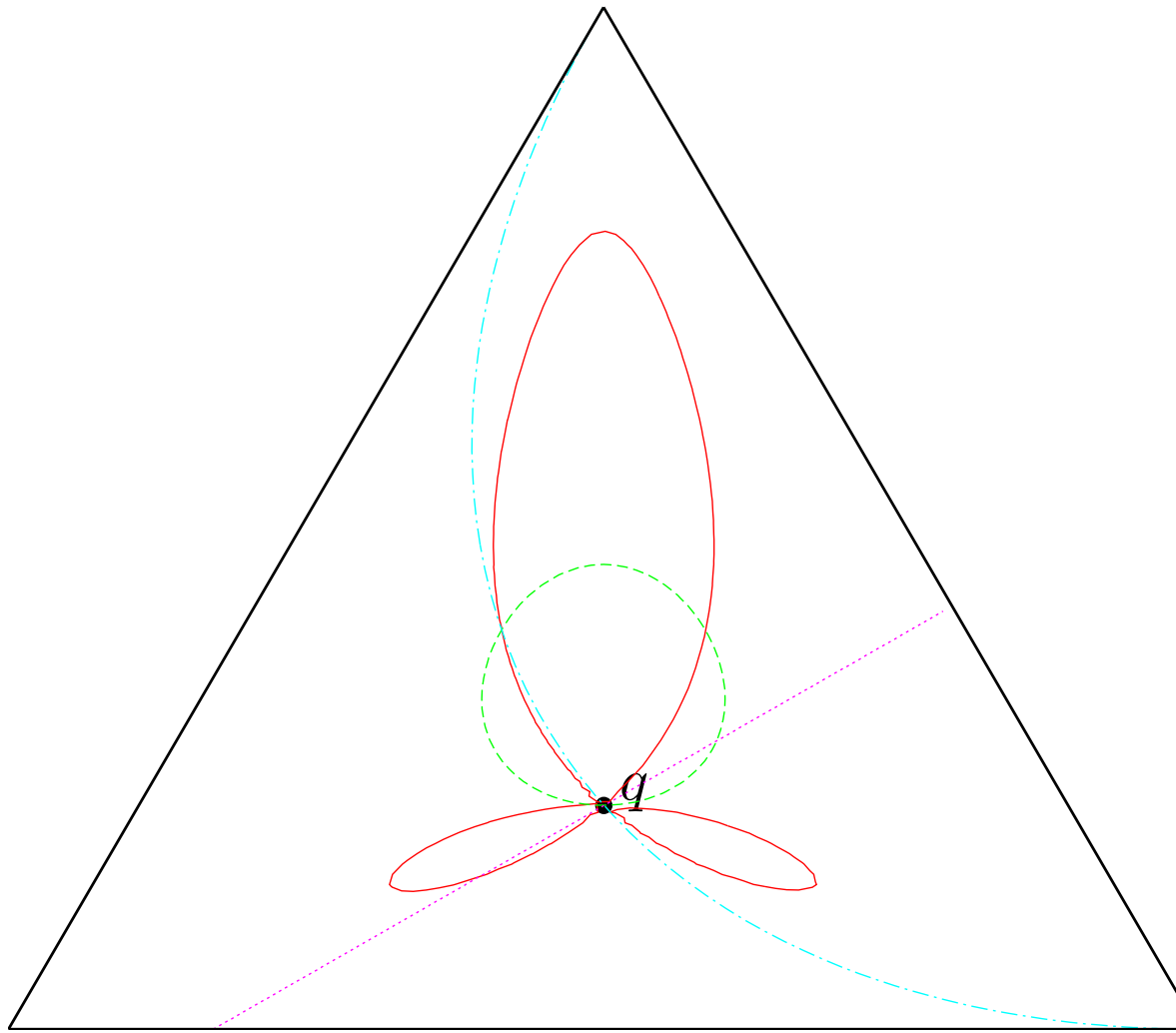

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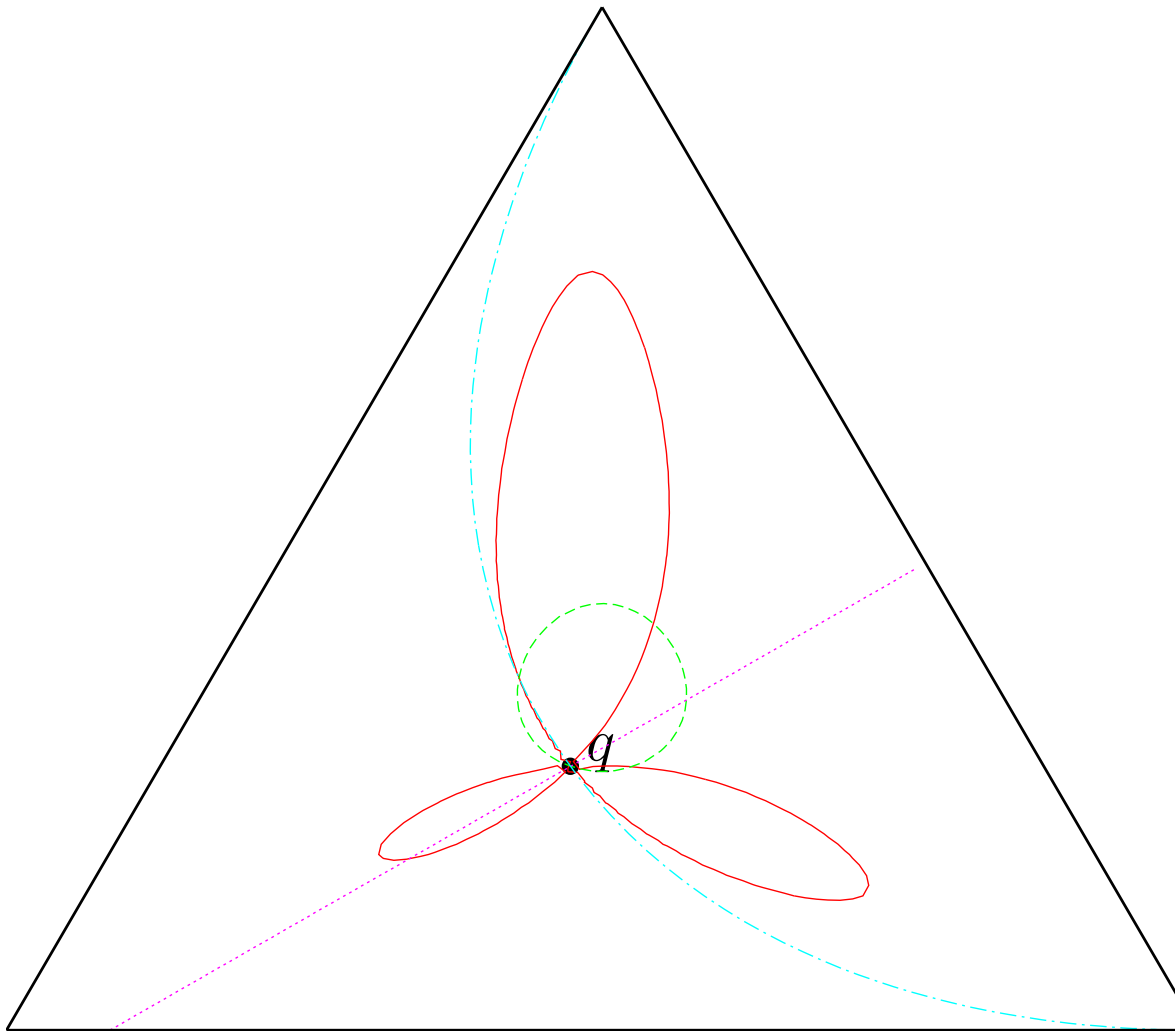

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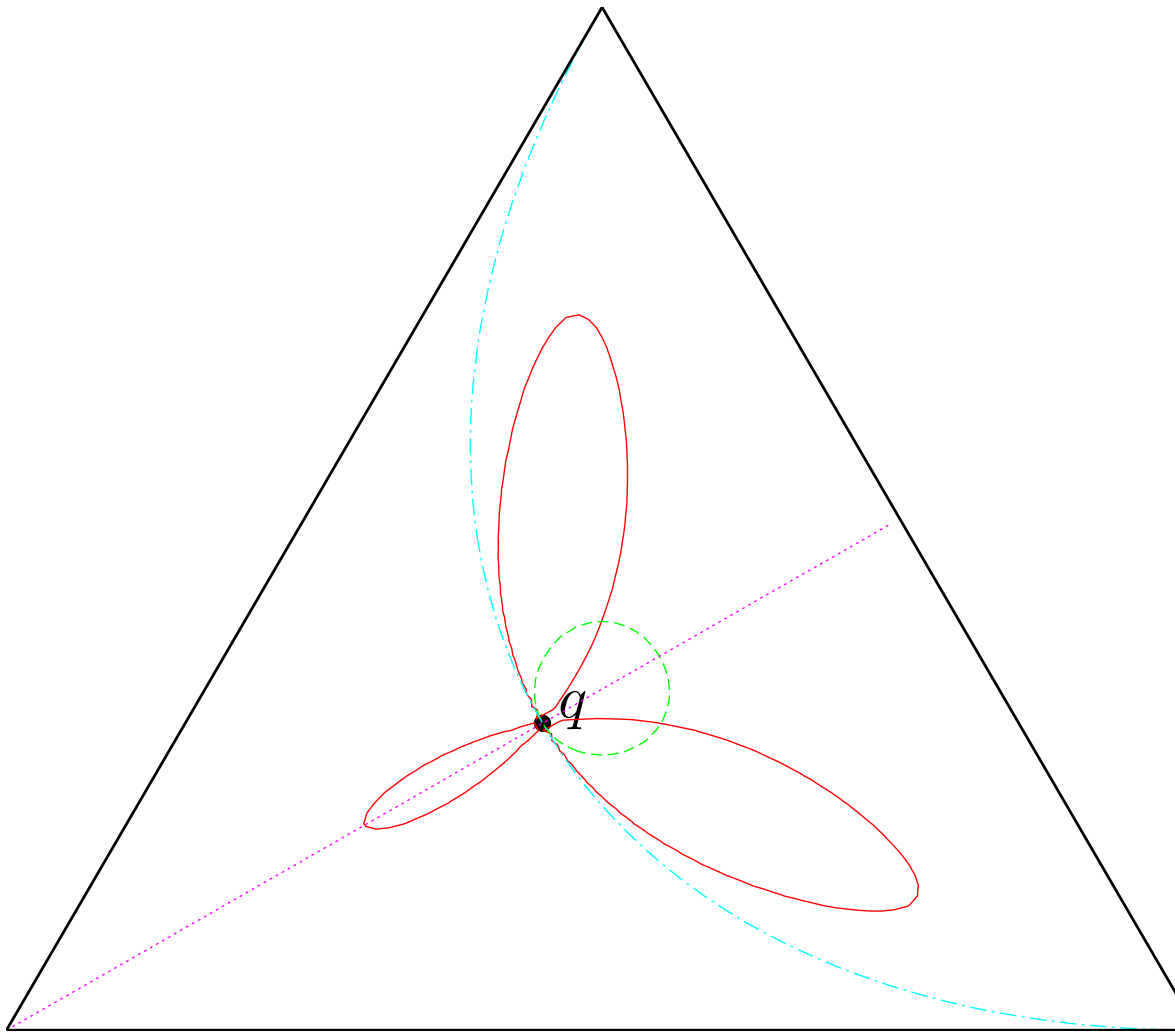

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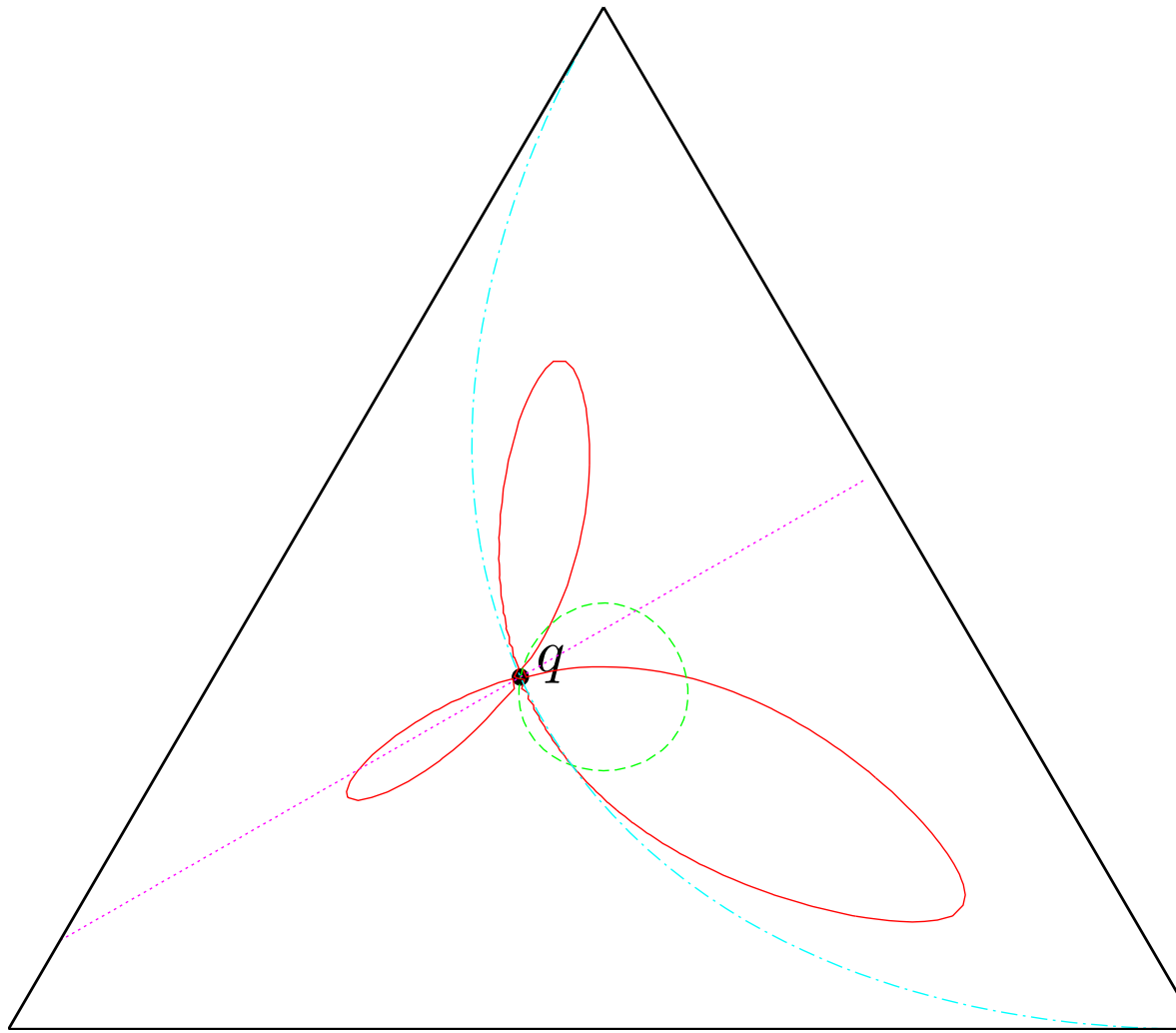

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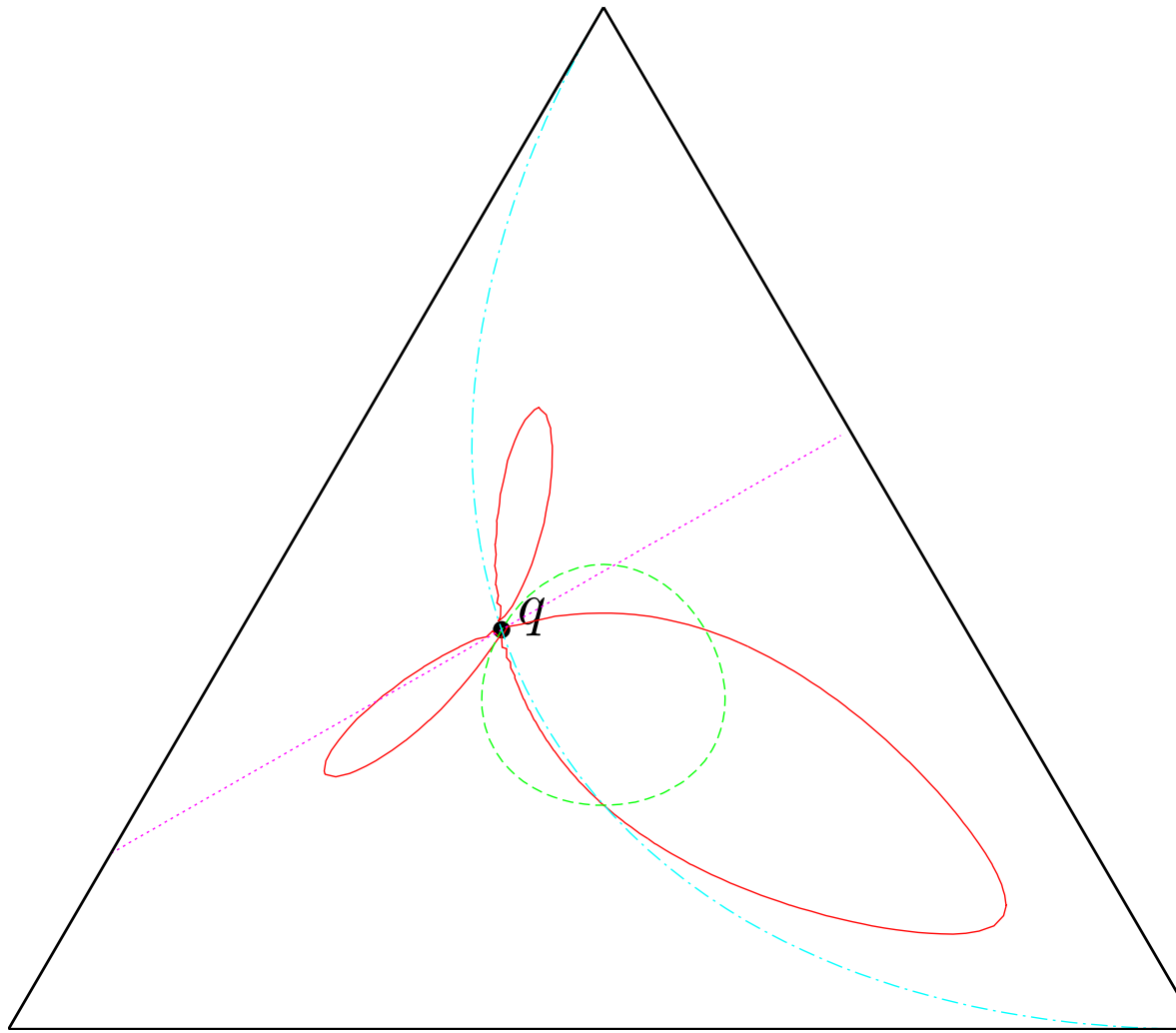

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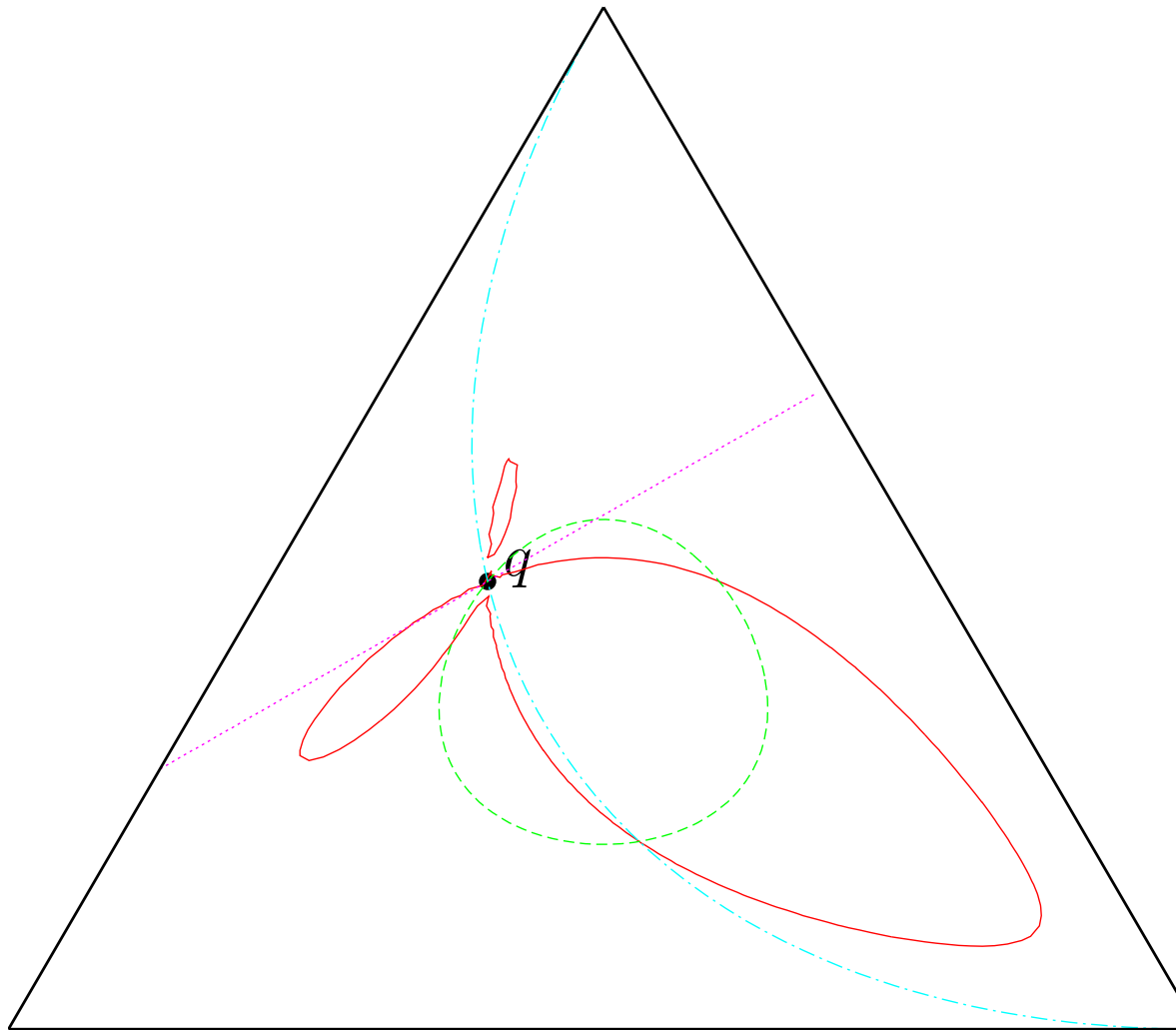

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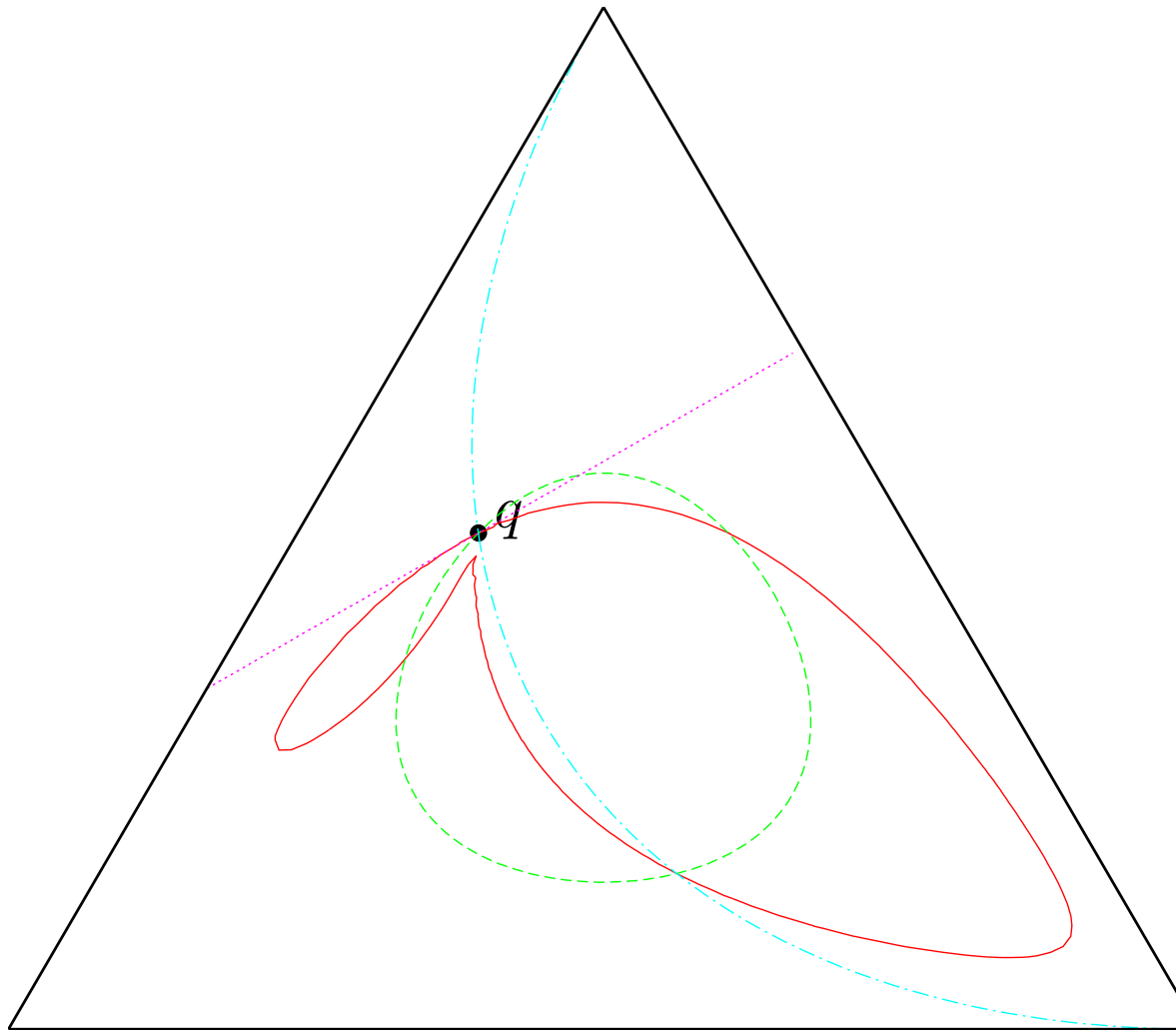

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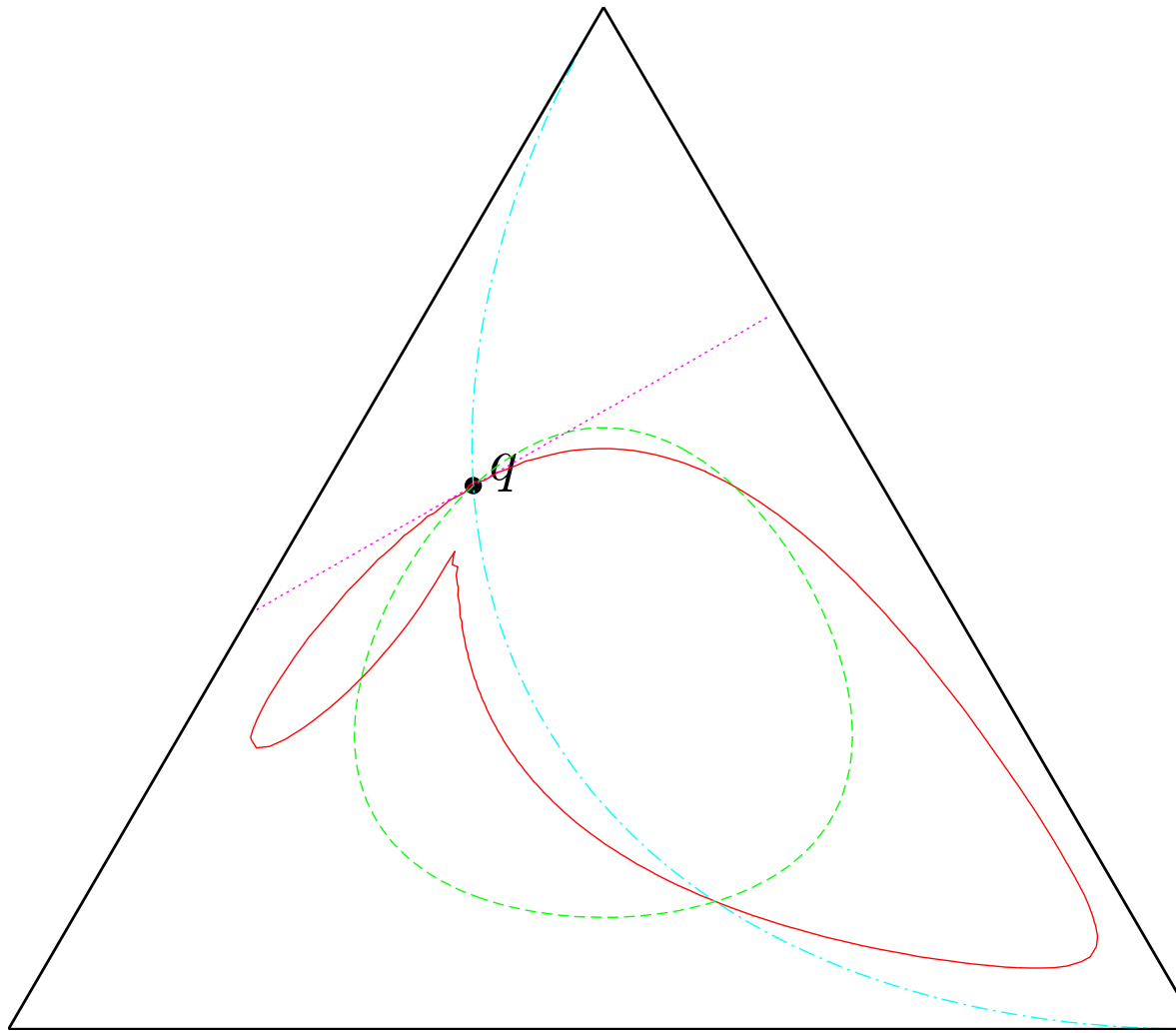

Geometry of optimal transition kernels

Theorem (Belavkin, 2013)

- *Let* $\{p(\beta)\}$ *be a family of* $p(\beta) \in \mathcal{P}(X \times Y)$

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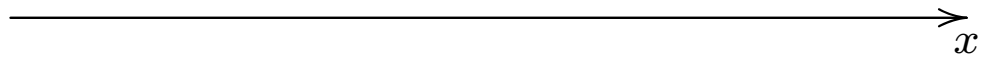
Example: Optimal Train Station



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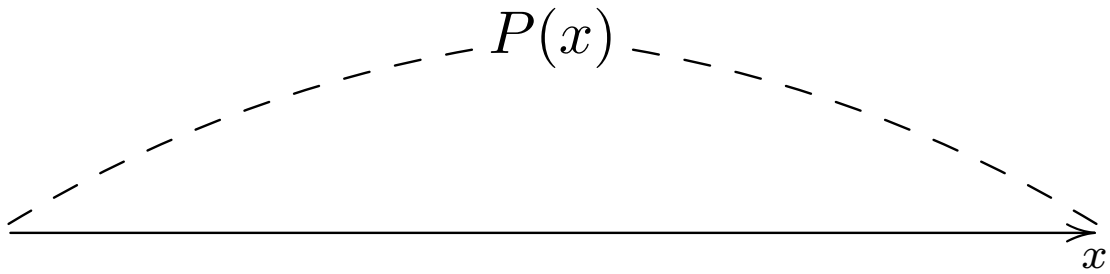

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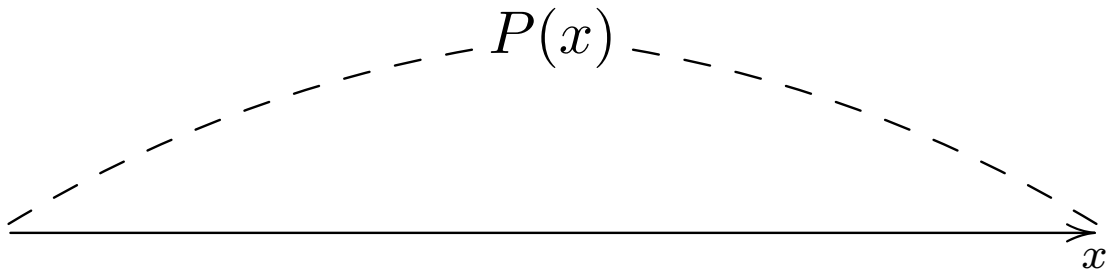

- Cost function $d(x, u) = |x - u|^2$

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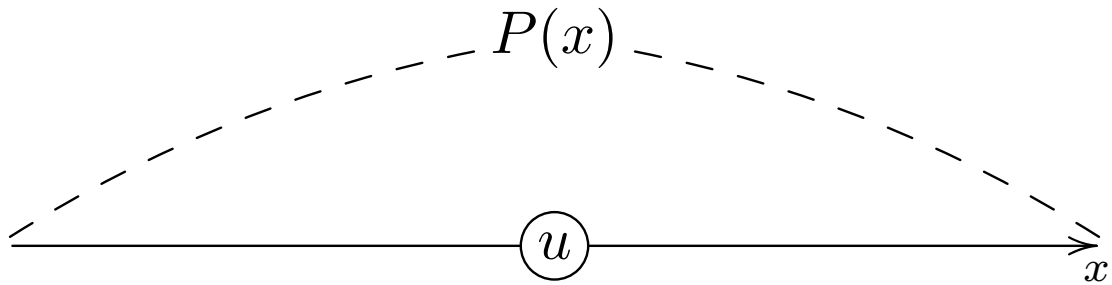

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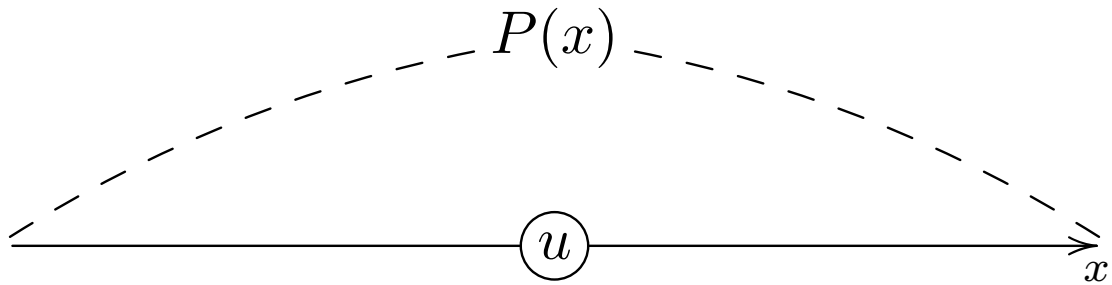

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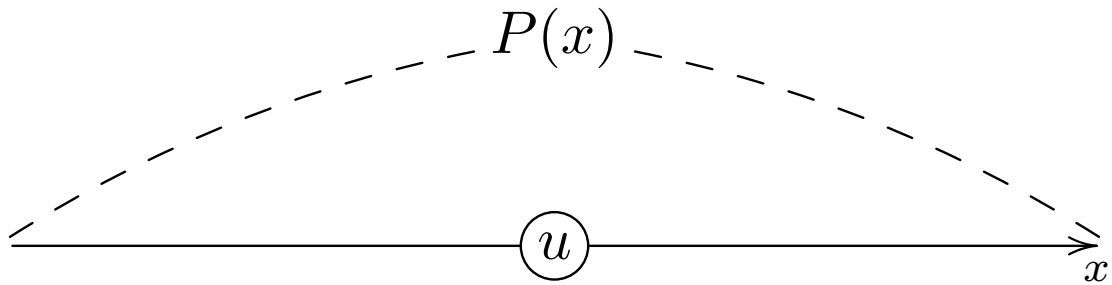

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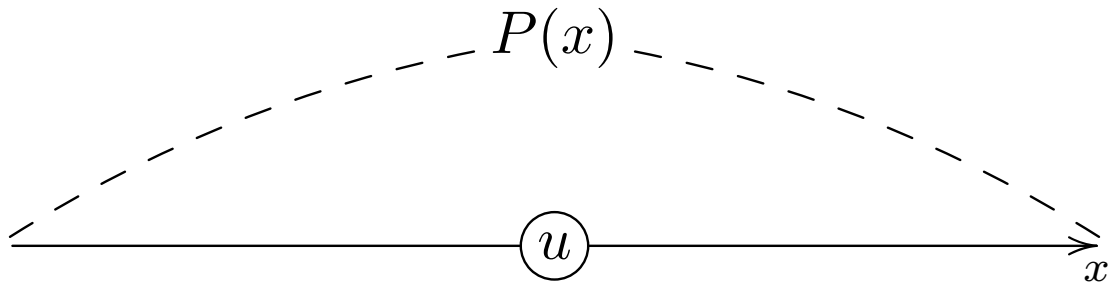

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Remark

If x has Cauchy distribution $dP(x) = [\pi(x^2+1)]^{-1} \, dx$, then $\mathbb{E}_P\{d(x,u)\} = \infty$ for any u!

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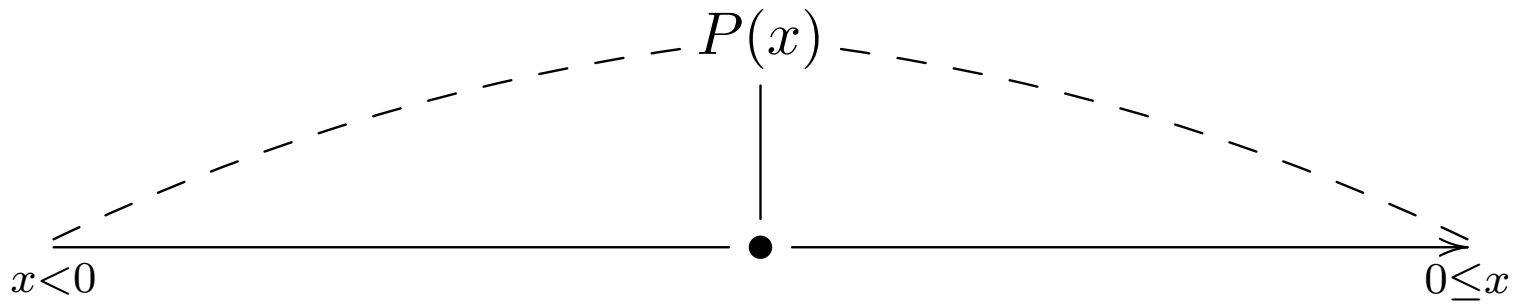

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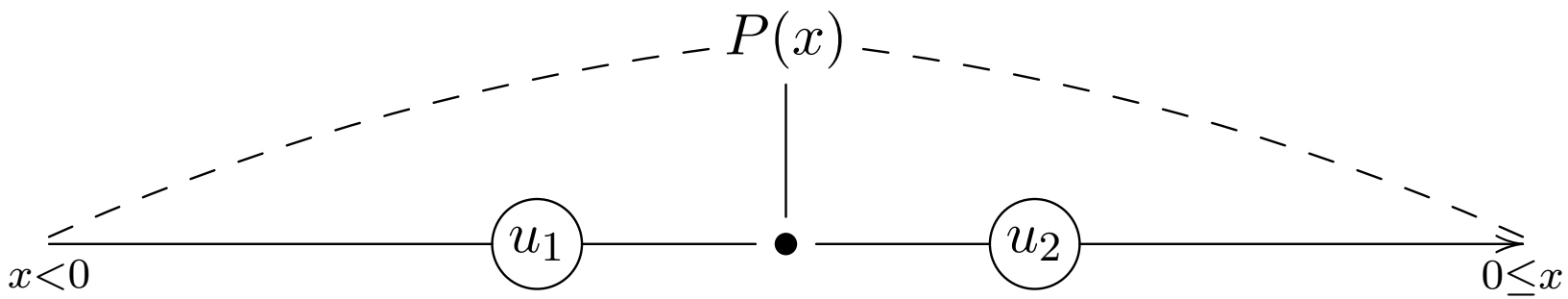

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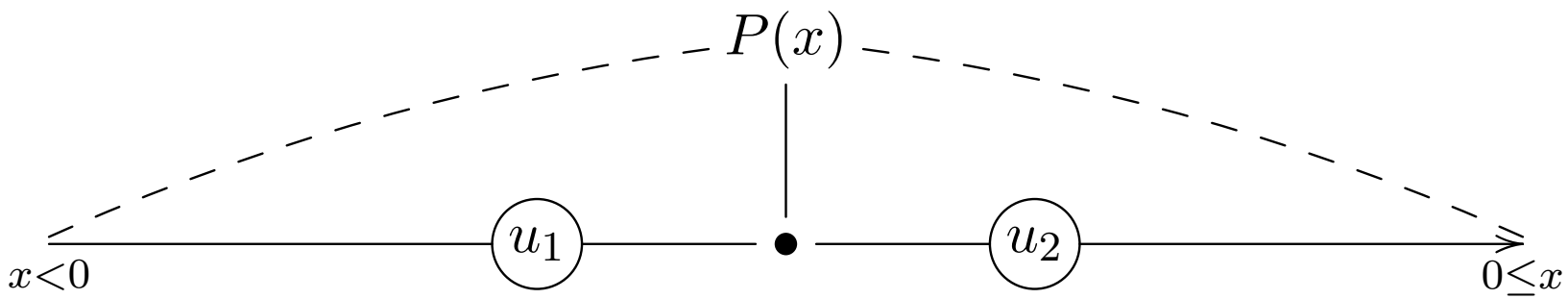

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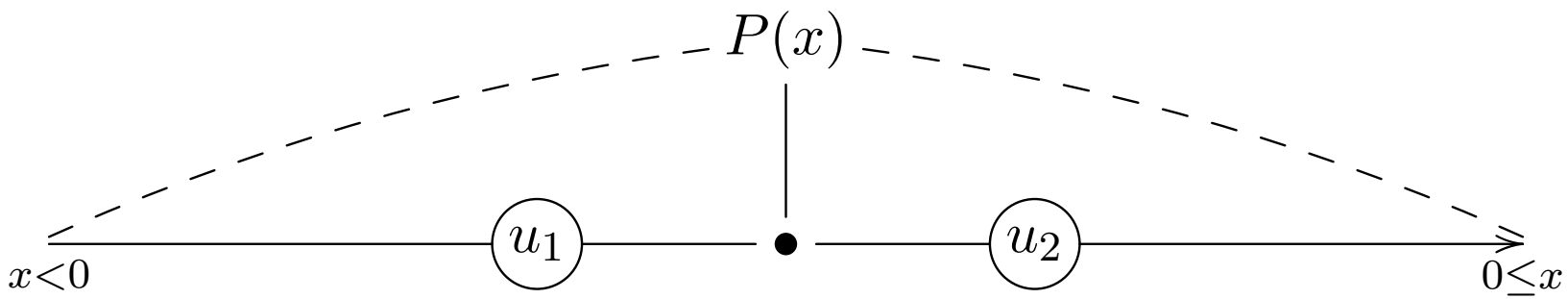

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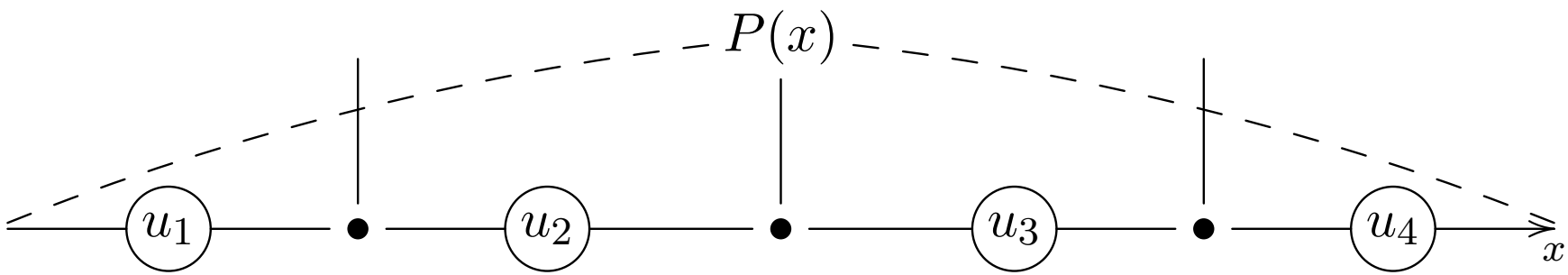
Example: Multiple Optimal Train Stations



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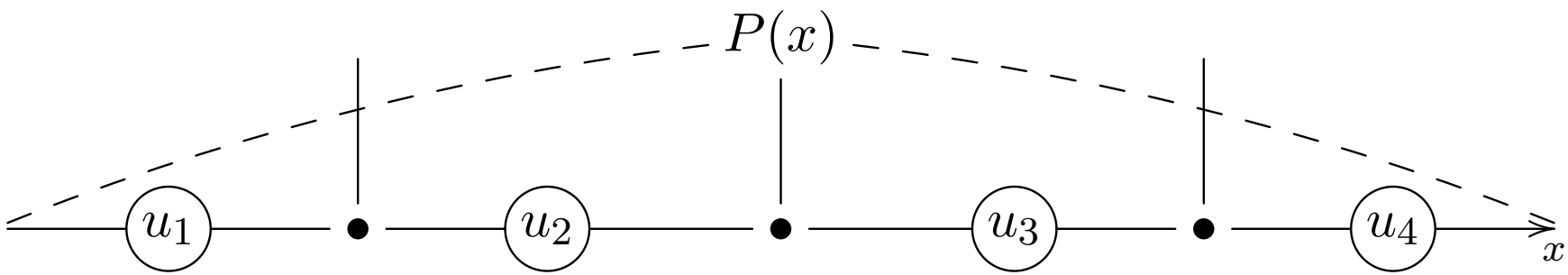

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Remark

If x has Cauchy distribution $dP(x) = [\pi(x^2+1)]^{-1} \, dx$, then
$\sum_{i=1}^{k} \mathbb{E}_P\{d(x,u_i) \mid y_i\} = \infty$ for any $k \in \mathbb{N}$ (i.e. any finite partition of $\mathbb{R}$)

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- Let X be a countable set of problems, and let $f(x)$ be a deterministic algorithm with time complexity $O(n^m)$.

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Remark

If $P(n) = n^{-(m+1)}/\zeta(m+1)$, *where* $\zeta(m+1) = \sum_{n \in \mathbb{N}} n^{-(m+1)}$, *then*

$$\mathbb{E}_P\{t\} = \zeta(1)/\zeta(m+1) = \infty$$

Example: Genetic search in Hamming Space

Example (Hamming metric)

Let $X = \{1, \dots, \alpha\}$ with Hamming metric d_H:

$$d(x, y) = l - \sum_{i=1}^{l} \delta_{x_i}(y_i)$$

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Example (Hamming metric)

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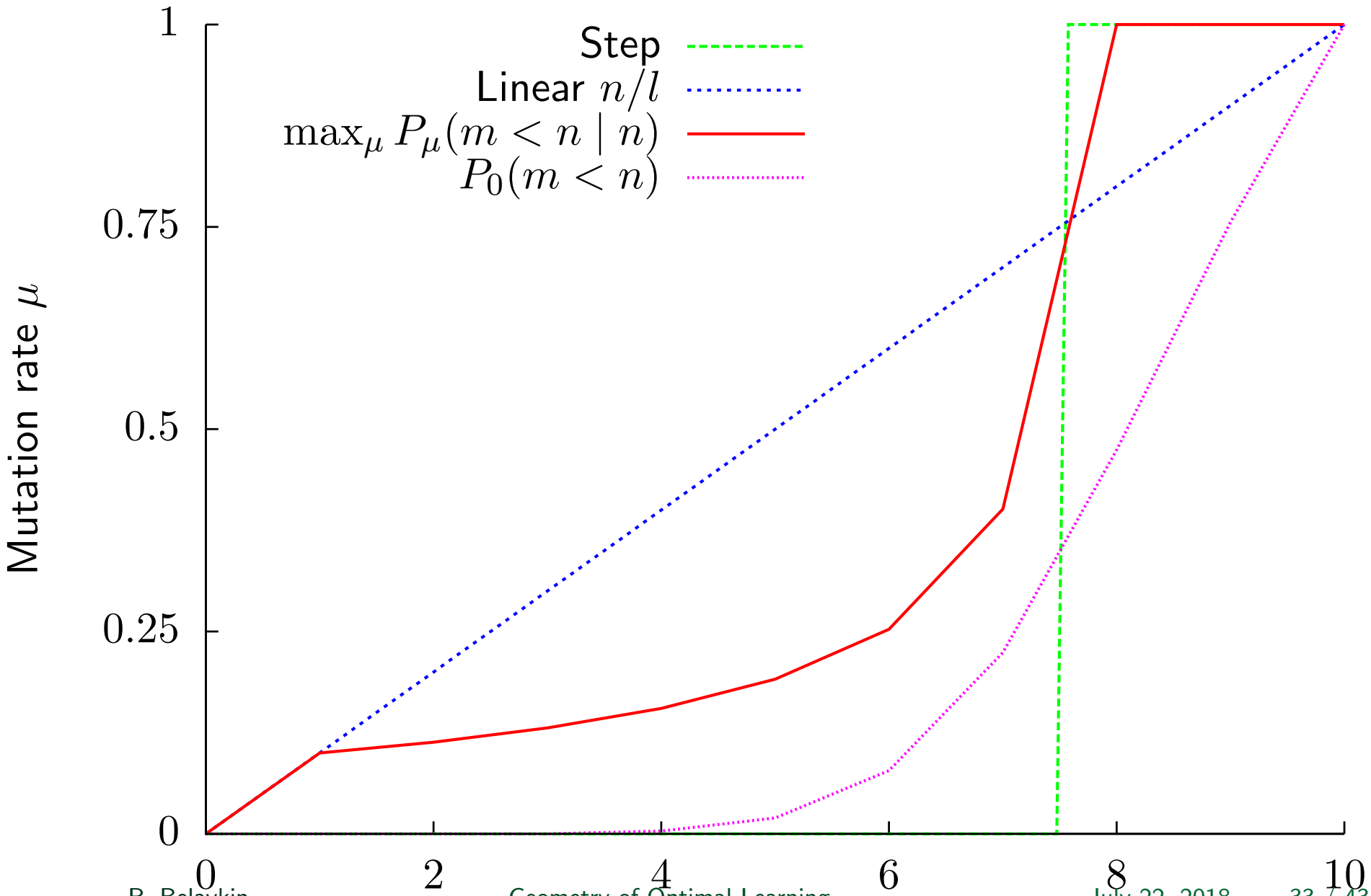
Solution

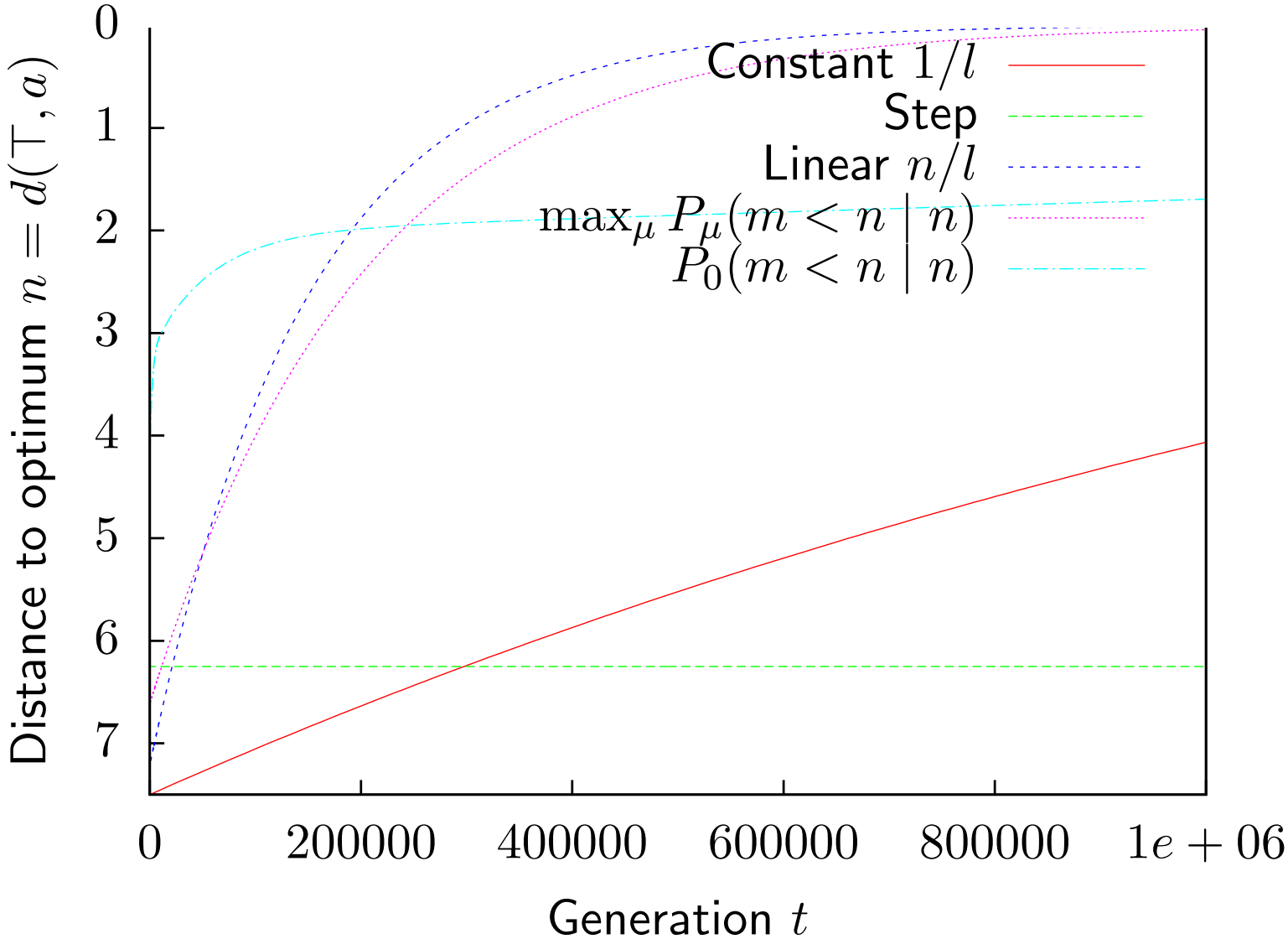
$$P_\beta(y \mid x) = \frac{e^{-\beta\, d(x,y)}}{[1 + (\alpha - 1)e^{-\beta}]^l} = \prod_{i=1}^{l} \frac{e^{-\beta\, \delta_{x_i}(y_i)}}{1 + (\alpha - 1)e^{-\beta}}$$

$\beta = \ln\left(\mu^{-1} - 1\right) + \ln(\alpha - 1)$, where $\mu = \upsilon/l$ is the mutation rate, defined by the constraint $\mathbb{E}\{d(x, y)\} \leq \upsilon$.

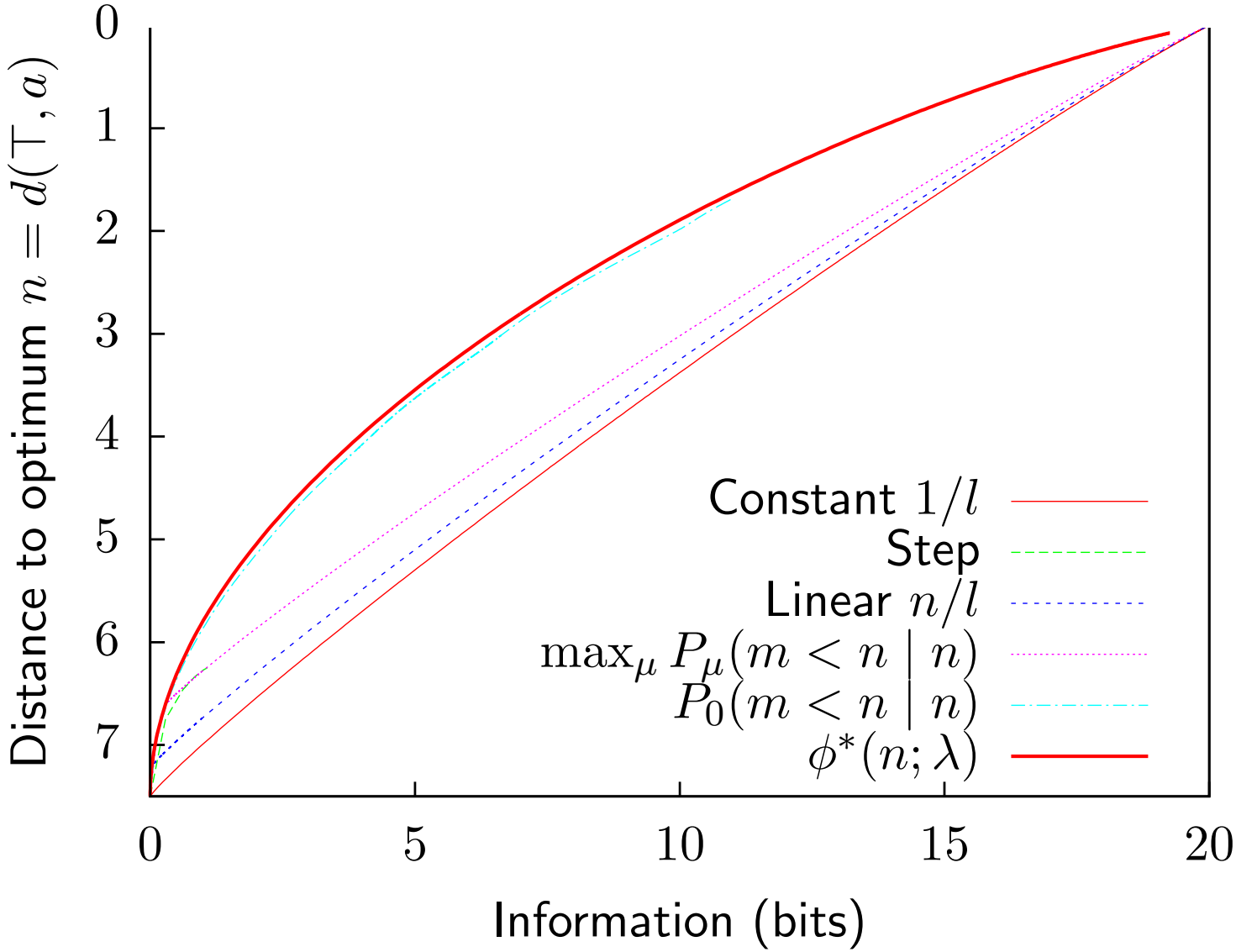
Mutation rate control functions in $\mathcal{H}_4^{10}$



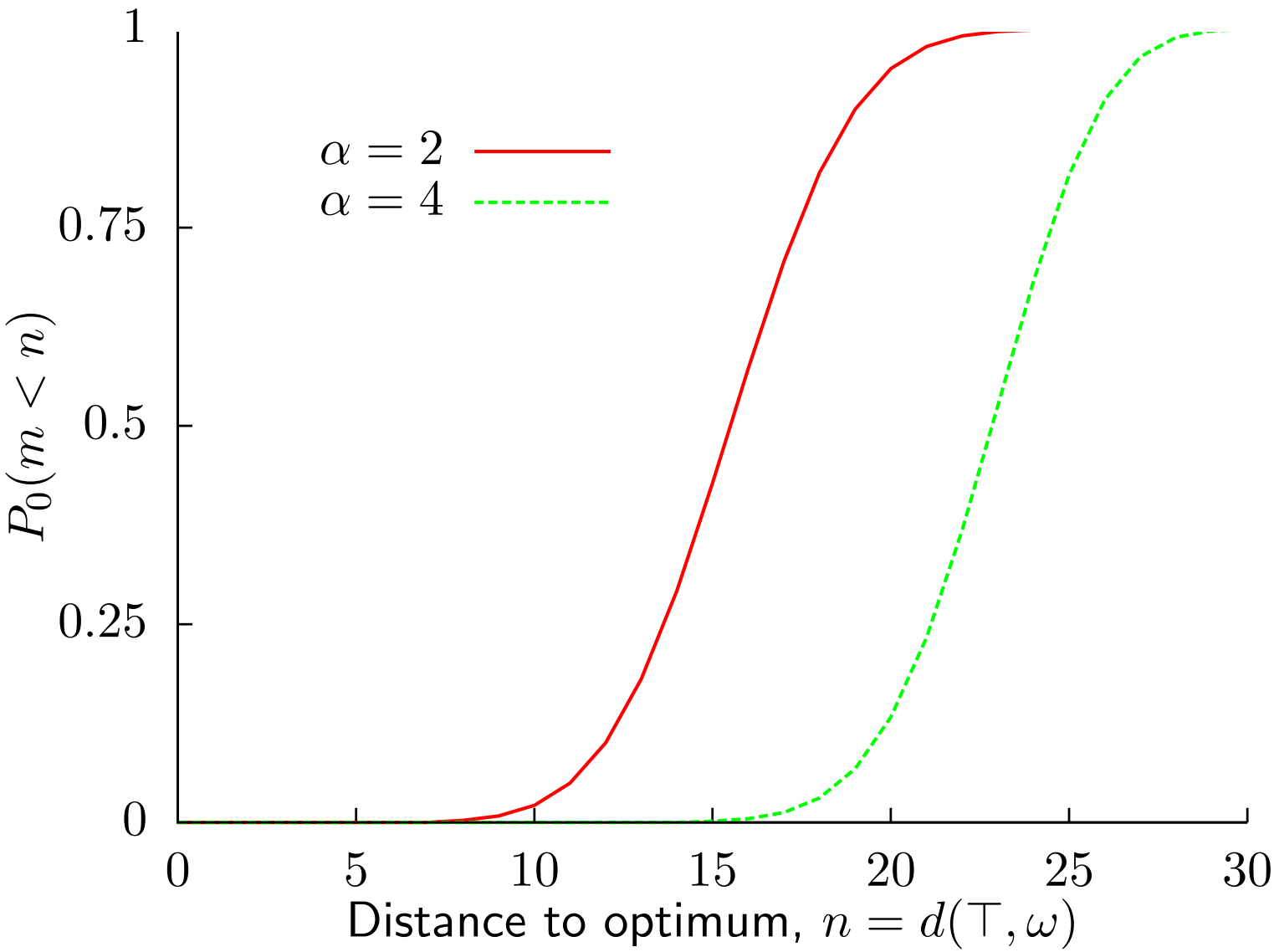
Expected Fitness in Time



Expected Fitness in Information



CDF Control


1
0.75
0.5
0.25
0
$P_0(m < n)$
$\alpha = 2$
$\alpha = 4$
0
5
10
15
20
25
30
Distance to optimum, $n = d(\top, \omega)$

Evolution as an Information Dynamic System

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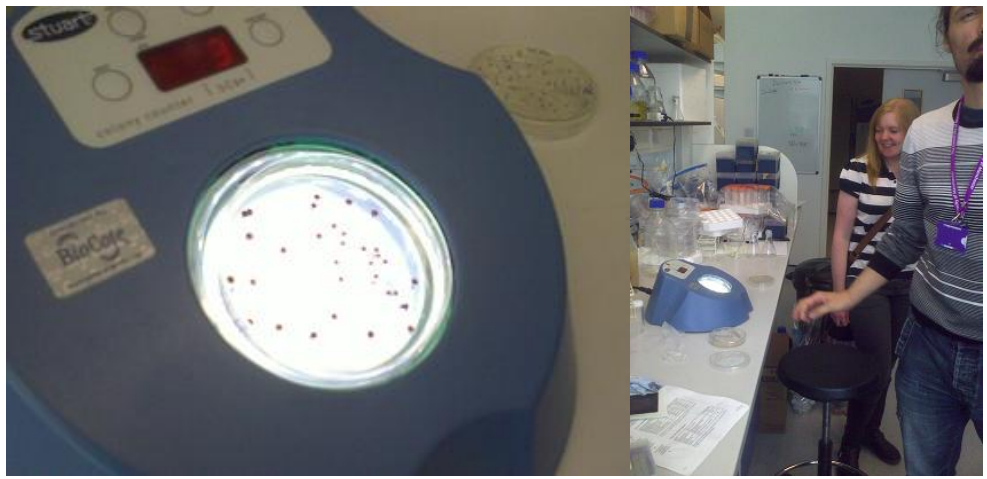
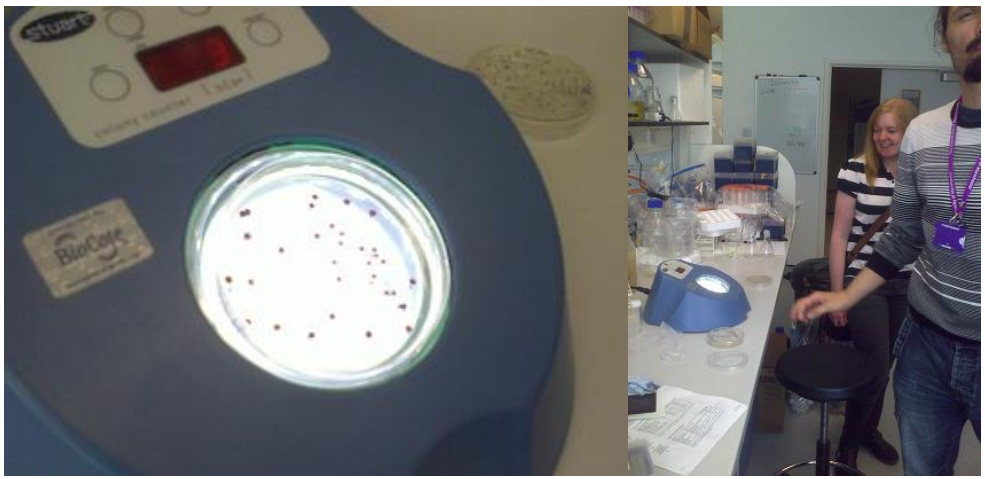
  University of Manchester : Chris Knight, Rok Krašovec & Danna Gifford

Mutation Rate Control in *E. coli*



- Used strains of *Escherichia coli* K-12 MG1665

Mutation Rate Control in *E. coli*



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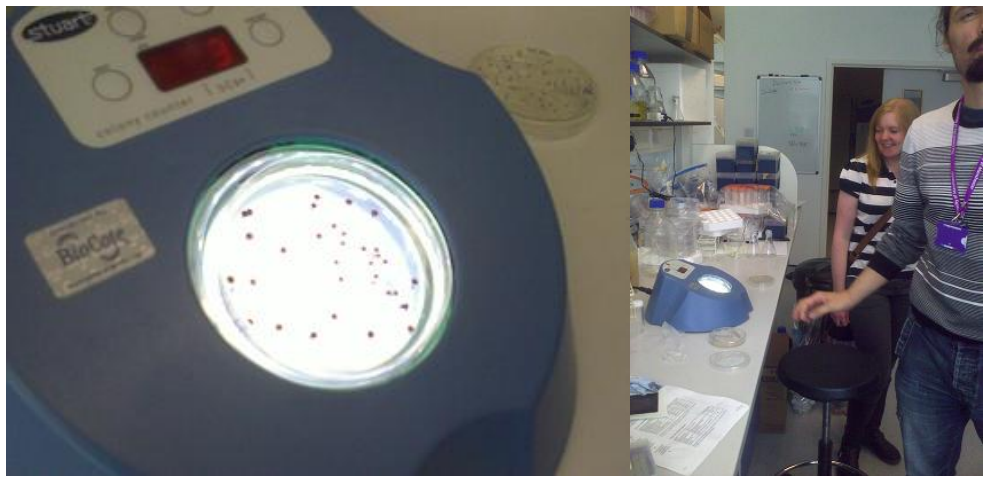
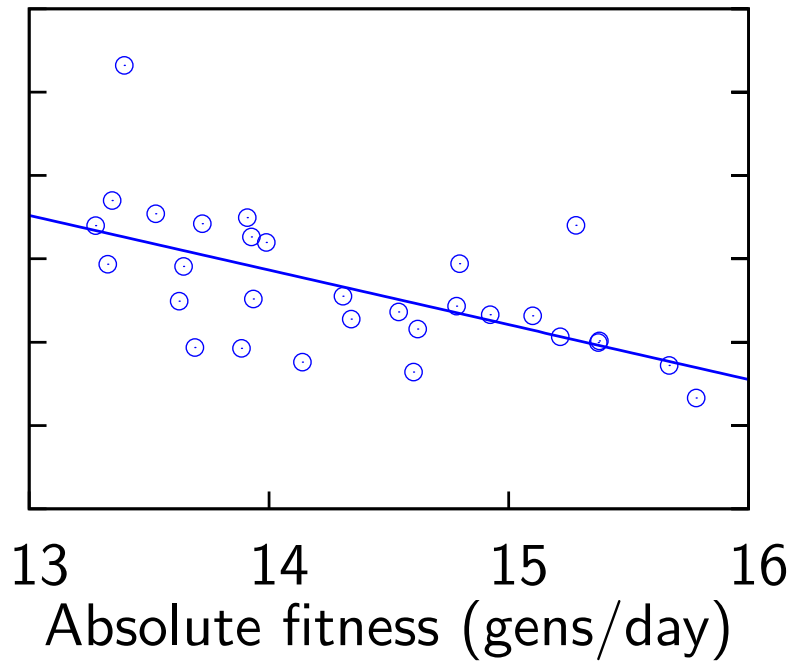
- Fluctuation test using media 50μg/ml of Rifamipicin

Mutation Rate Control in *E. coli*



- Used strains of *Escherichia coli* K-12 MG1665
- Fluctuation test using media 50μg/ml of Rifamipicin
- Estimated mutation rates μ in *E.coli* strains grown in Davis minimal medium with different amount of glucose.

Experimental Results (Krašovec et al., 2014)



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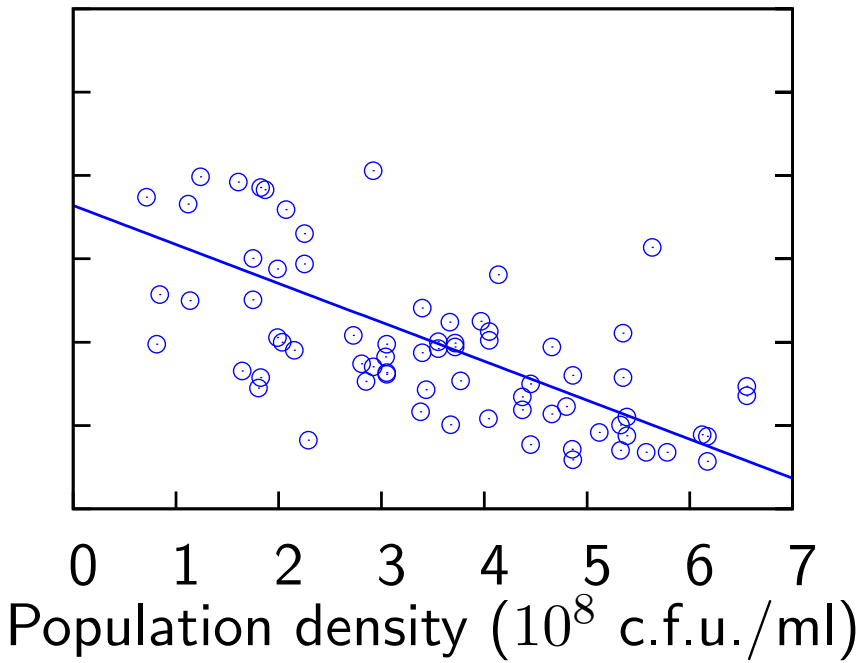

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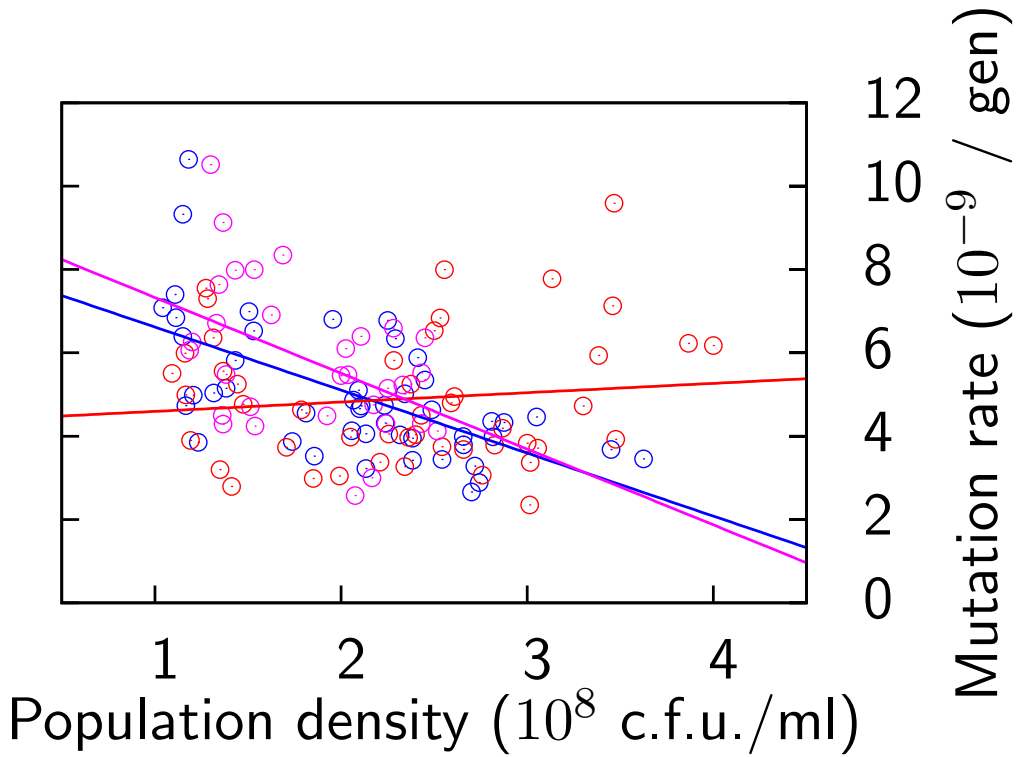

- Strong relationship between μ and density of cells ($p < .0001$).

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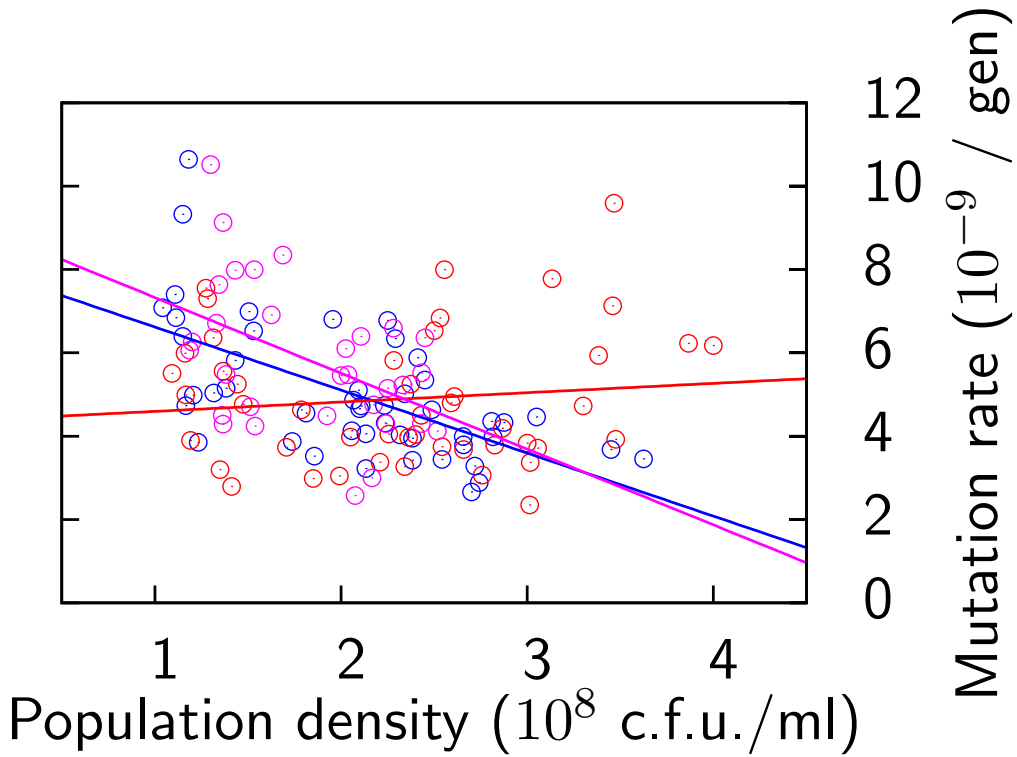

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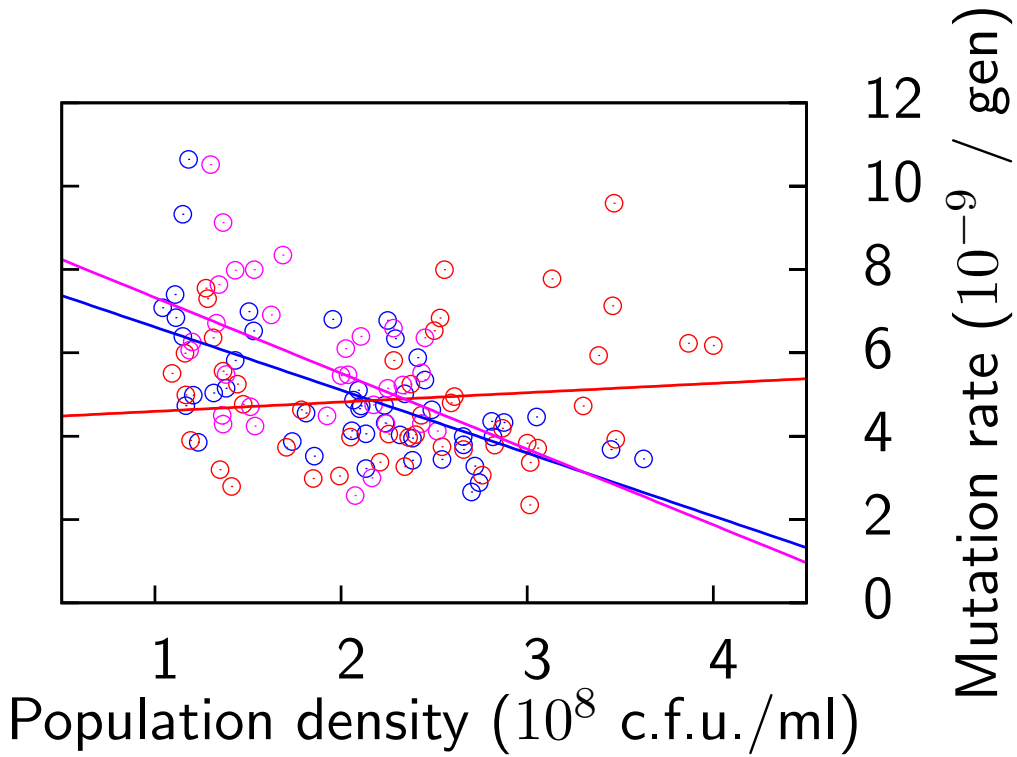

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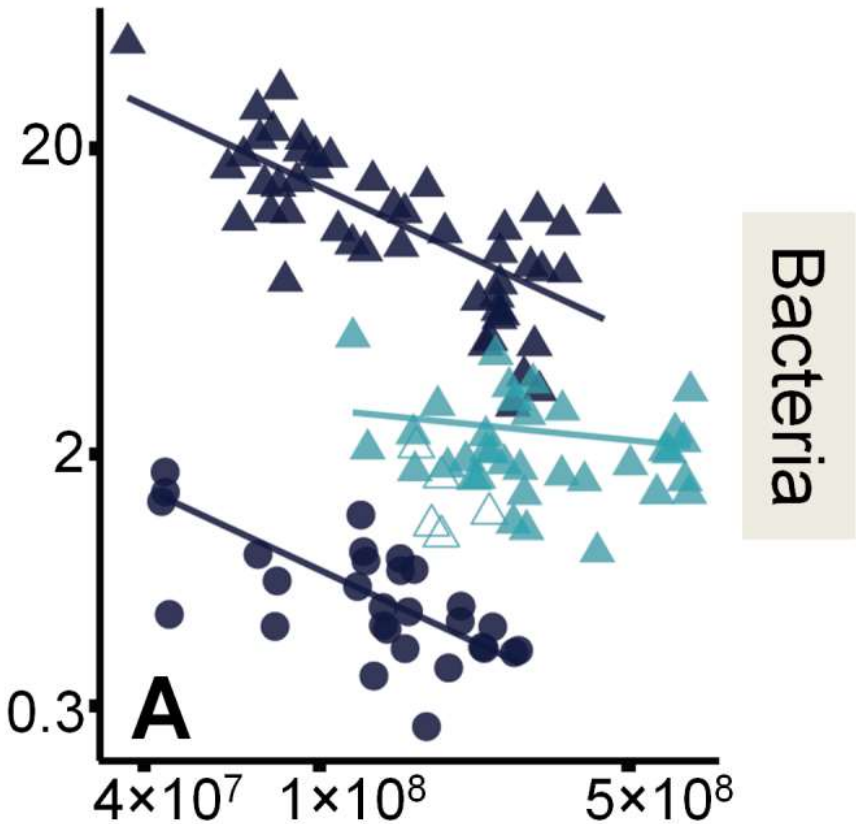
Plastic mutation rates in bacteria (Krašovec et al., 2017)



- *rifampicin* (triangles)

Krašovec, R., Richards, H., Gifford, D. R., Hatcher, C., Faulkner, K. J., Belavkin, R. V., Channon, A., Aston, E., McBain, A. J., Knight, C. G. (2017). Spontaneous mutation rate is a plastic trait associated with population density across domains of life. *PLoS Biology*, 15:8.

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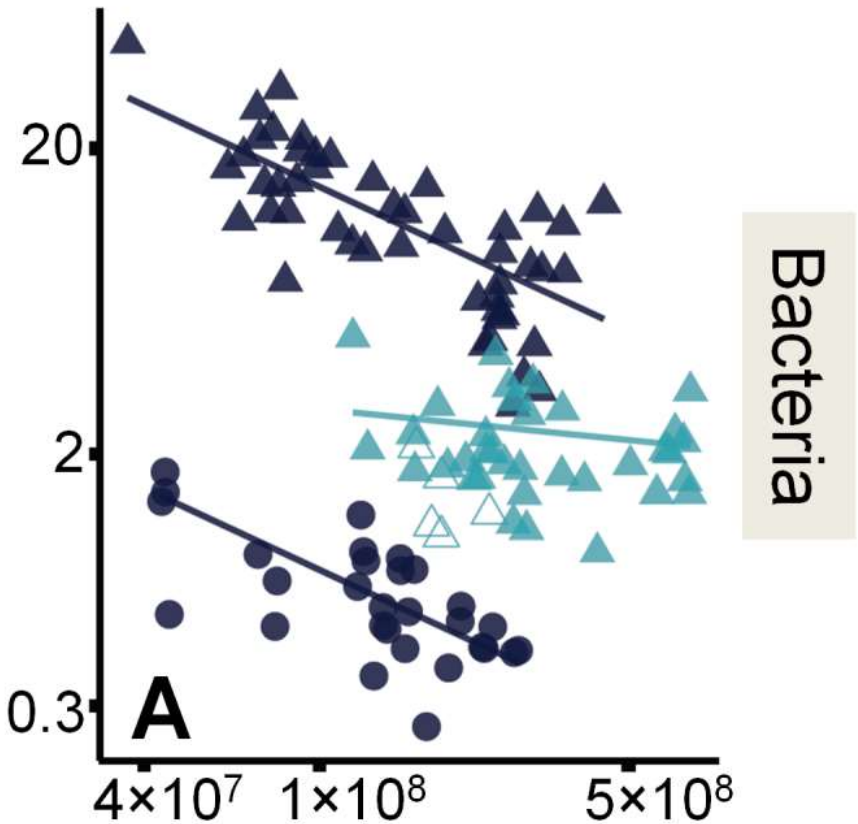

- *rifampicin* (triangles)
- *nalidixic acid* in *E.coli* (dark circles) and in *P. aeruginosa* (light circles)

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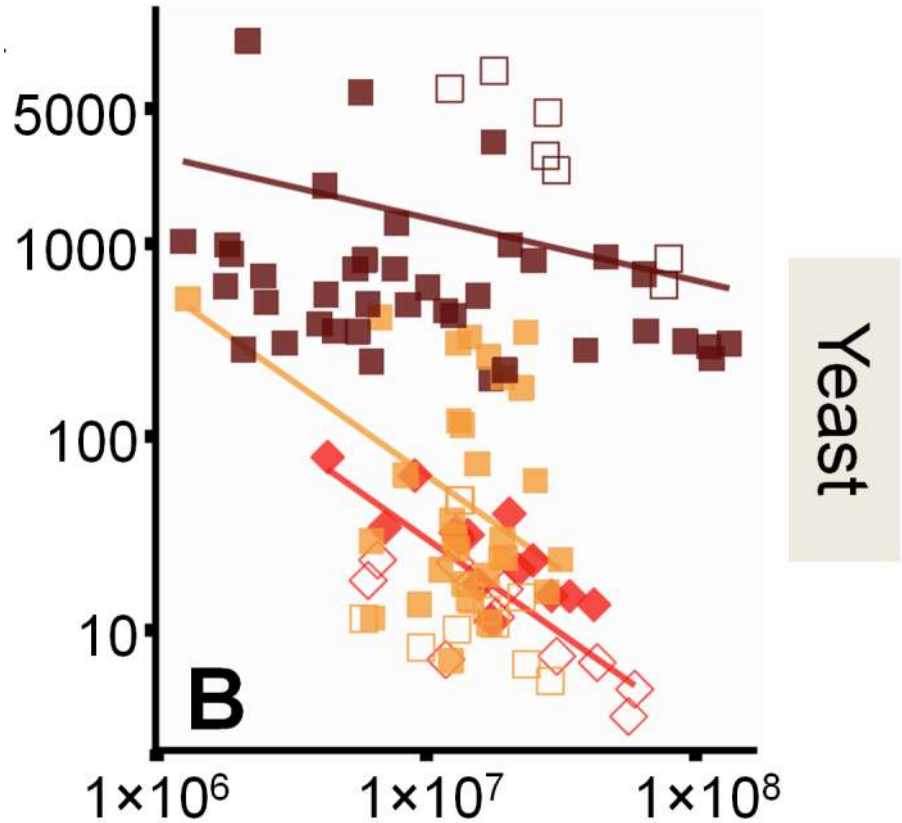

- *hygromycin B* (squares) in *S. cerevisiae*

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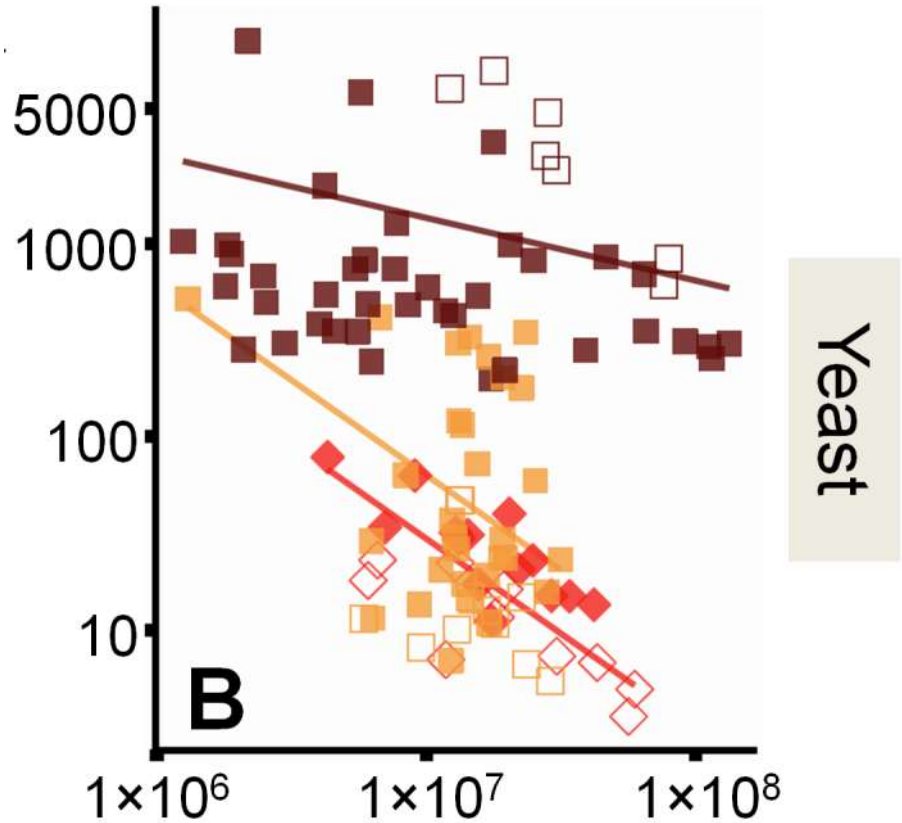

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- *5-FOA* (diamonds)

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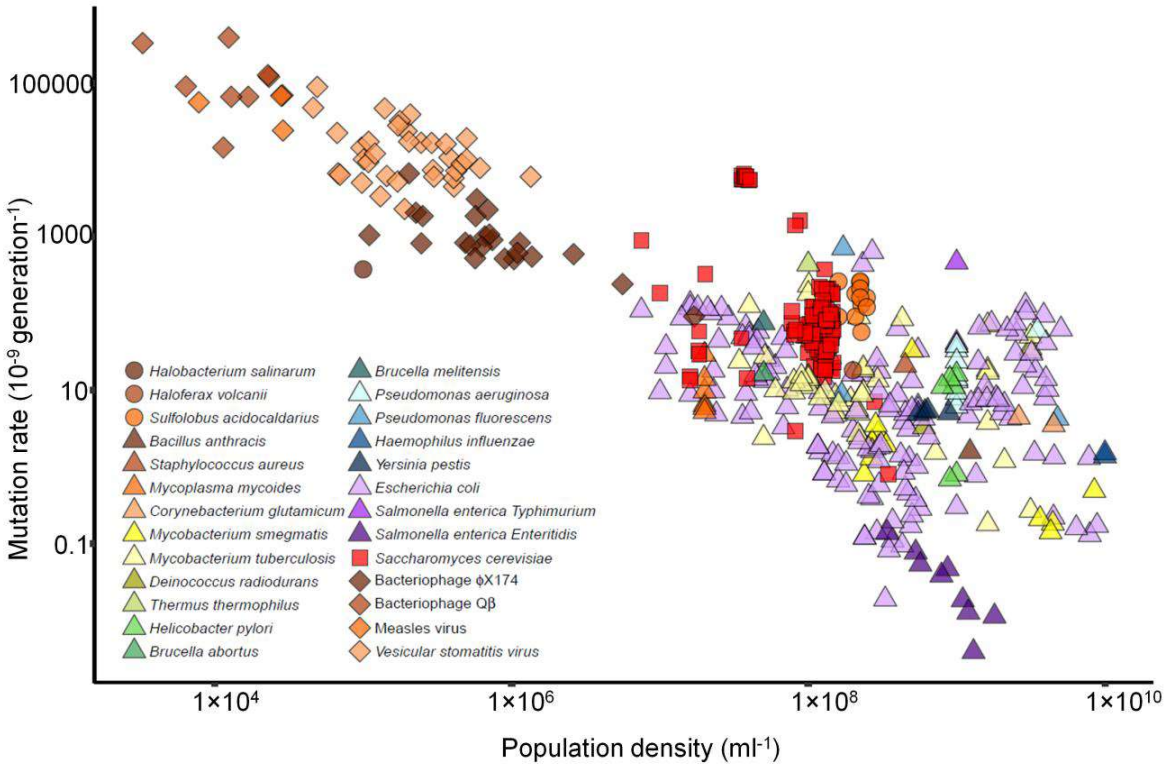
Plastic rates in all domains of life (Kraševec et al., 2017)



>70 years of published data (1943–2016), 67 studies, 26 species.

Kraševec, R., Richards, H., Gifford, D. R., Hatcher, C., Faulkner, K. J., Belavkin, R. V., Channon, A., Aston, E., McBain, A. J., Knight, C. G. (2017). Spontaneous mutation rate is a plastic trait associated with population density across domains of life. *PLoS Biology*, 15:8.

Optimal learning

First variational problem

Value of information

Iterative algorithms as Markov morphisms

Optimal transition kernels

Optimal control of mutation rate

Belavkin, R. V. (2013). Optimal measures and Markov transition kernels. *Journal of Global Optimization*, *55*, 387–416.

Krašovec, R., Belavkin, R. V., Aston, J. A. D., Channon, A., Aston, E., Rash, B. M., et al. (2014, April). Mutation rate plasticity in rifampicin resistance depends on escherichia coli cell-cell interactions. *Nature Communications*, *5*(3742).

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Shannon, C. E. (1948, July and October). A mathematical theory of communication. *Bell System Technical Journal*, *27*, 379–423 and 623–656.

Stratonovich, R. L. (1965). On value of information. *Izvestiya of USSR Academy of Sciences, Technical Cybernetics*, *5*, 3–12. (In Russian)

Stratonovich, R. L. (1968). Is there a theory for a synthesis of optimal adaptive, self-learning and self-organising systems? *Automatics and Telemechanics*, *1*. (In Russian)