Sequence Models

Ramin Hasani

ACDL 2024

Agenda

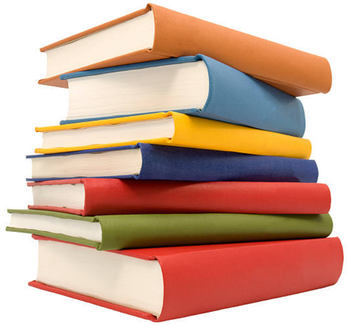
- Introduction, motivation, prior approaches
- Linear state space models (SSMs) overview
- S4, convolutions, parameterization
- S5, diagonalization, parallel scans
- S6/Mamba/Hawk/Griffin, data-dependent dynamics
- Challenges and opportunities

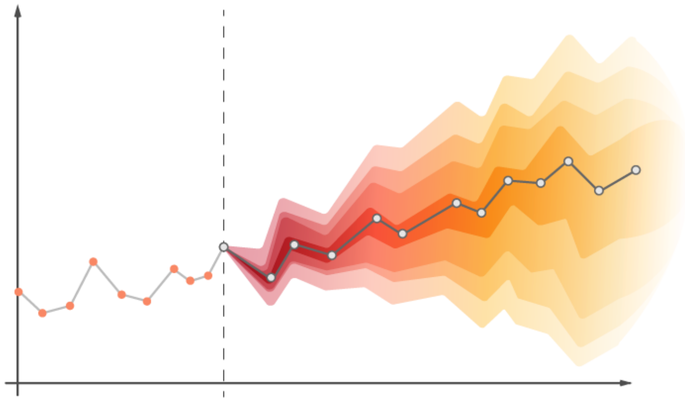
Agenda

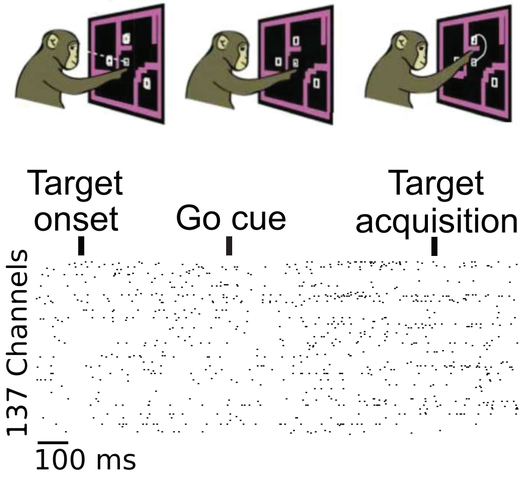
- **Introduction, motivation, prior approaches**
- Linear state space models (SSMs) overview
- S4, convolutions, parameterization
- S5, diagonalization, parallel scans
- S6/Mamba/Hawk/Griffin, data-dependent dynamics
- Challenges and opportunities

Motivation: Efficiently modeling long sequences

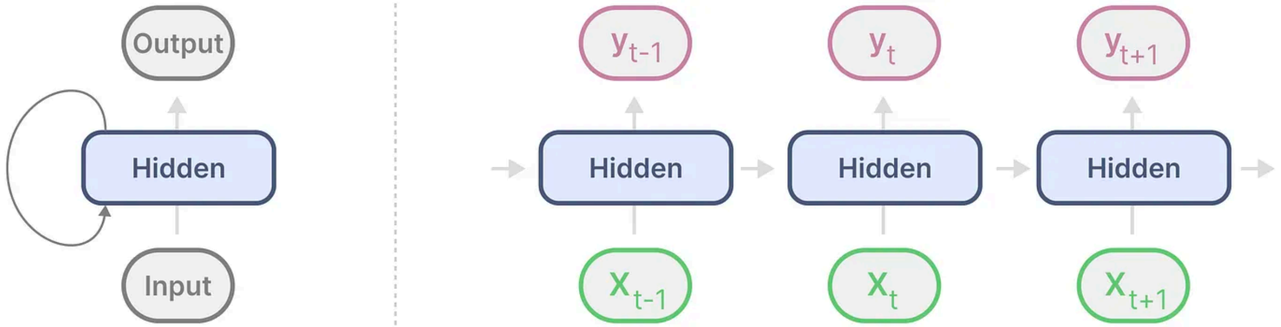
Applications: text, audio, forecasting, neuroscience, images, videos







Nonlinear Recurrent Neural Networks

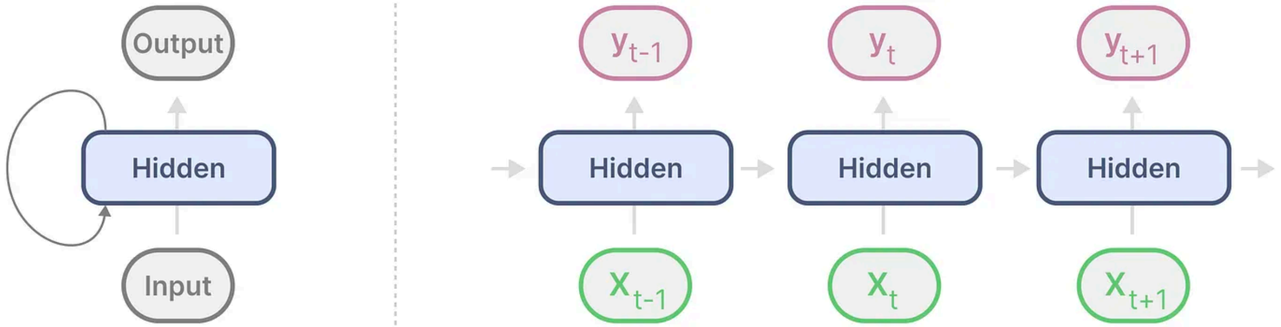


RNN

RNN
(Unfolded)

Image Source: https://maartengrootendorst.substack.com/p/a-visual-guide-to-mamba-and-state

Nonlinear Recurrent Neural Networks



RNN

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	Recurrent Neural Networks (RNNs)
Parallelizable training	X

Inherently sequential forward and backward pass

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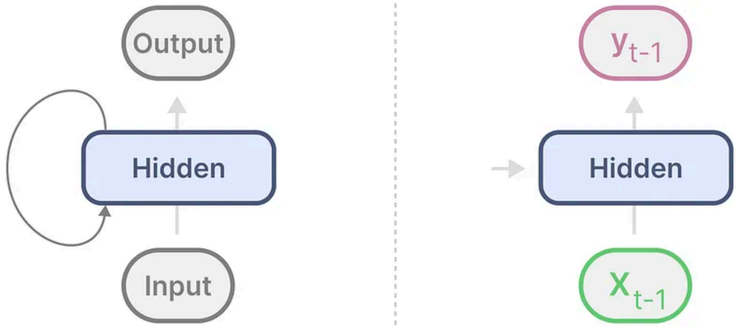
Nonlinear Recurrent Neural Networks



RNN

RNN
(Unfolded)

	Recurrent Neural Networks (RNNs)
Parallelizable training	✗
Fast autoregressive generation	✓

Inherently sequential forward and backward pass

Constant time and space required to perform single step of generation

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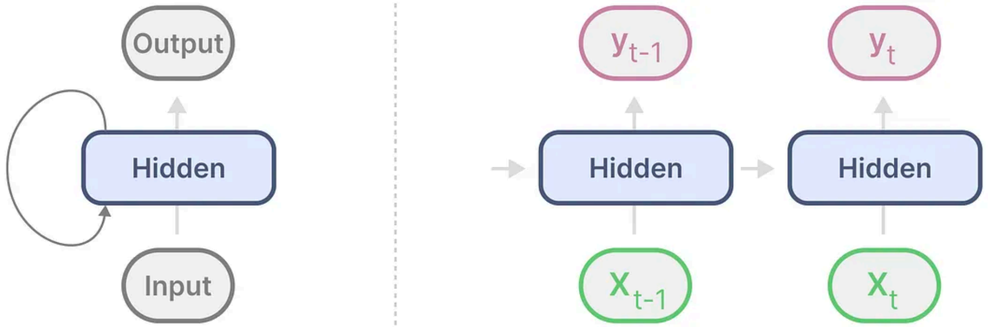
Nonlinear Recurrent Neural Networks



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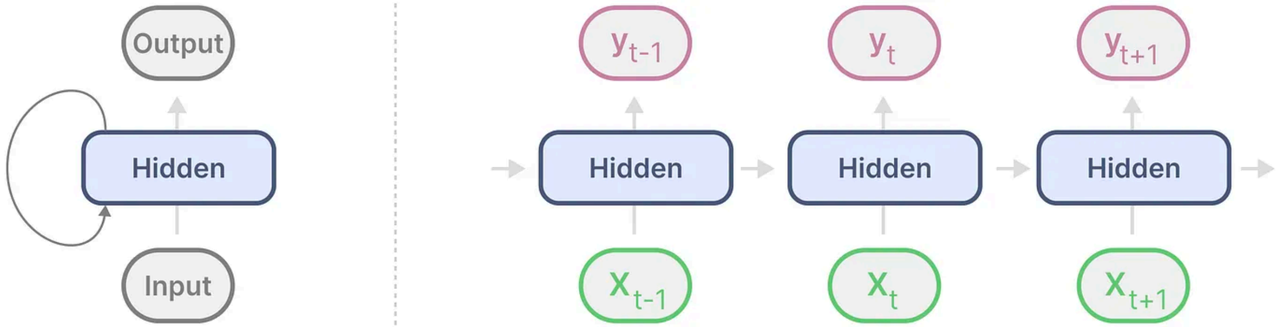

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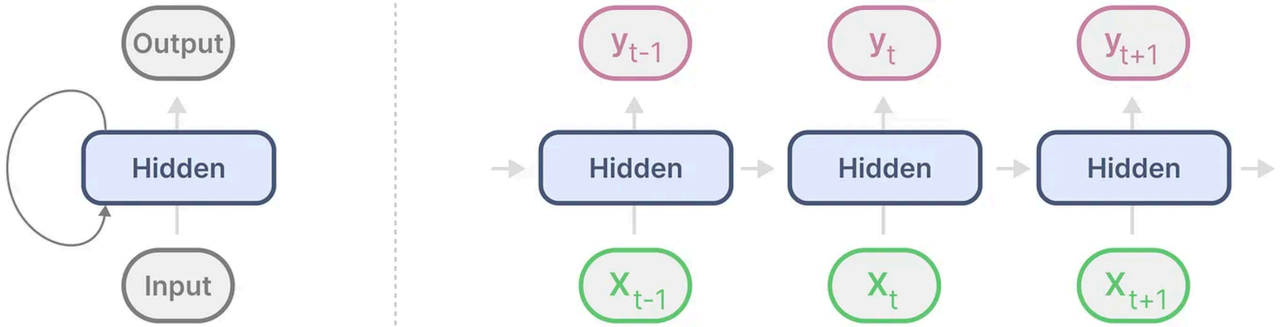


RNN

RNN
(Unfolded)

	Recurrent Neural Networks (RNNs)
Parallelizable training	✗
Fast autoregressive generation	✓
Avoid vanishing gradients	✗

Inherently sequential forward and backward pass

Constant time and space required to perform single step of generation

Difficult to train to retain information from the past due to this

Image Source: https://maartengrootendorst.substack.com/p/a-visual-guide-to-mamba-and-state

Attention

Training

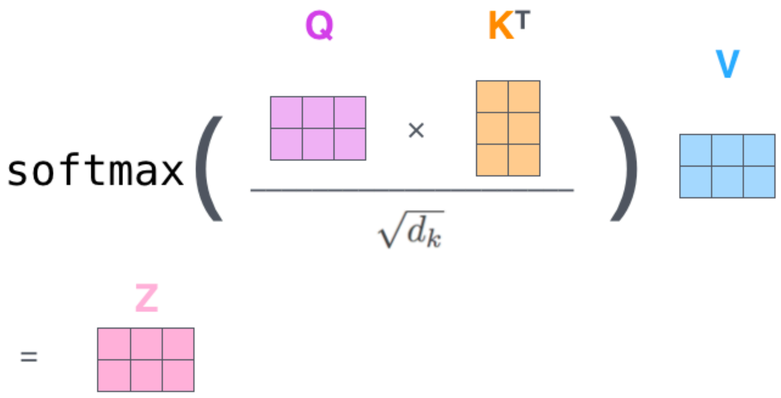
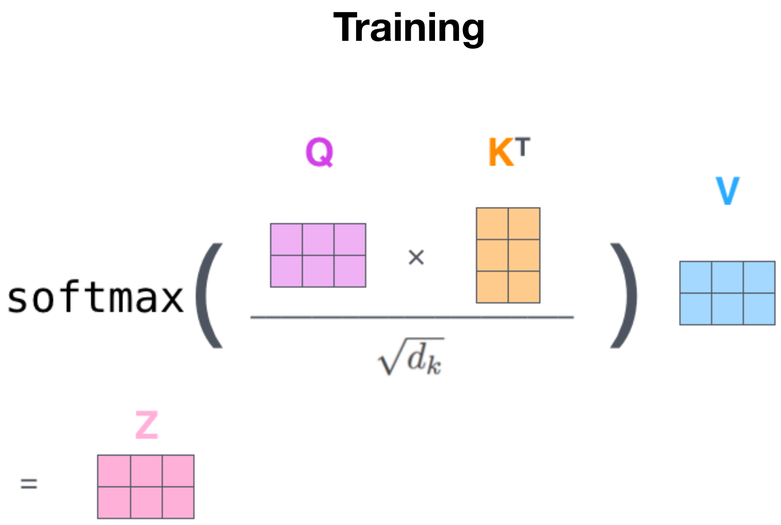


Image Source: https://jalammar.github.io/illustrated-transformer/https://jalammar.github.io/illustrated-transformer/, Oren 2024: https://arxiv.org/abs/2401.06104

Attention



	Attention
Parallelizable training	✓

Matrix multiplications, modern hardware (GPUs/TPUs) is highly optimized for this (though quadratic complexity)

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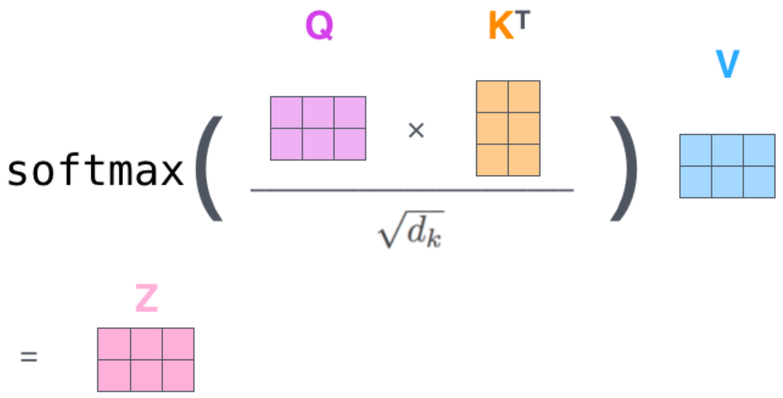

	Attention
Parallelizable training	✓
Fast autoregressive generation	X

Matrix multiplications, modern hardware (GPUs/TPUs) is highly optimized for this (though quadratic complexity)

Again quadratic complexity, have to compare to all past keys and values each step
(growing “state” size, aka KV cache)

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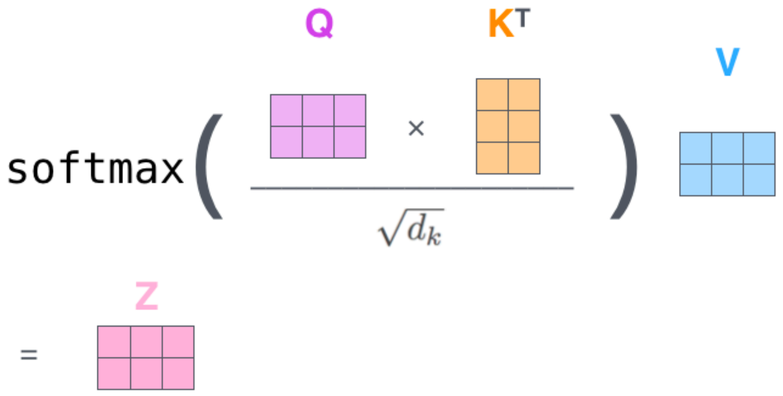
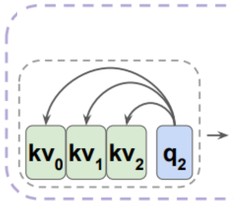


Autoregressive Generation



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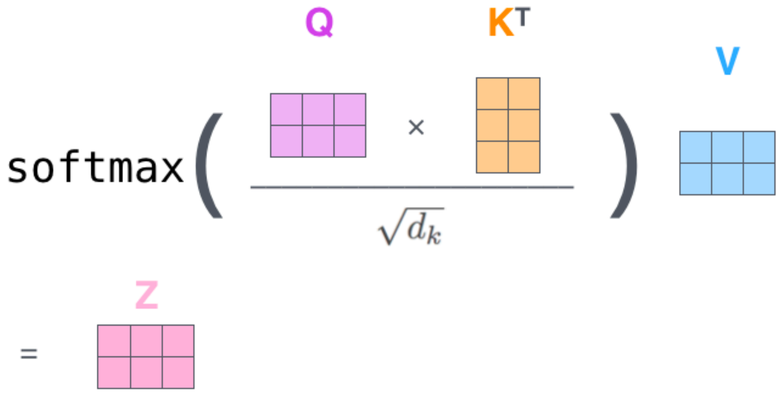

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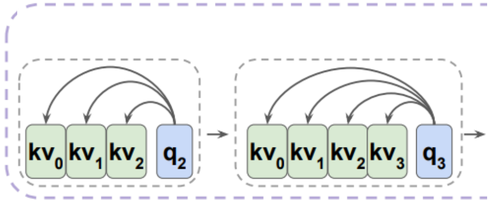

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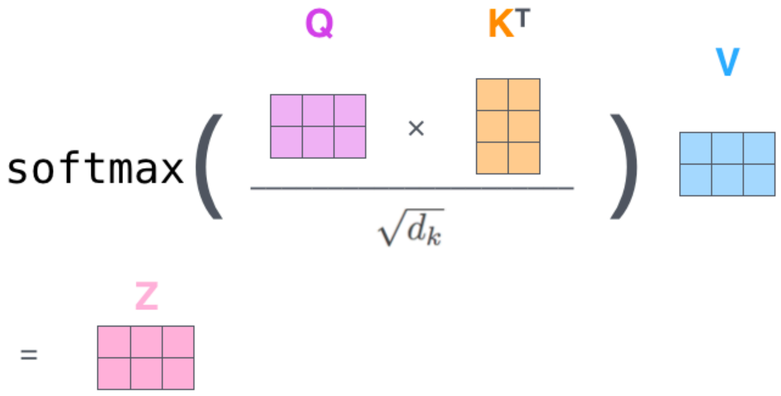
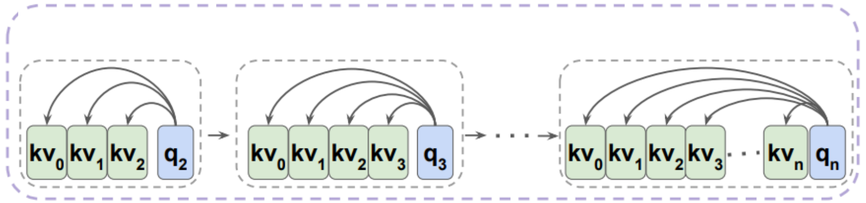


Autoregressive Generation



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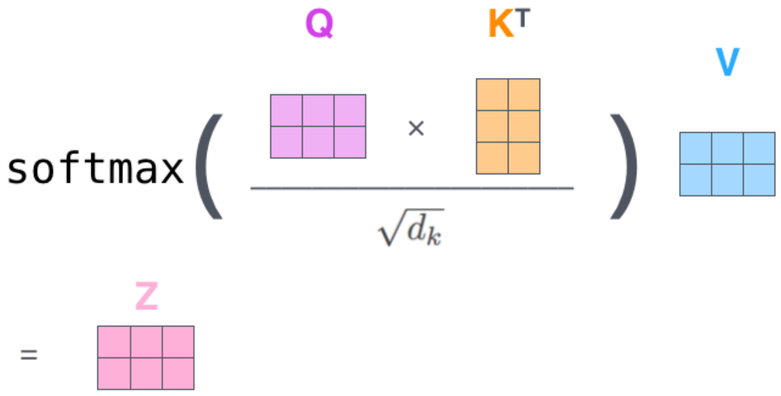

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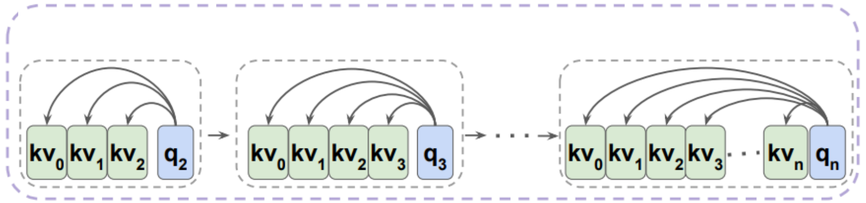

	Attention
Parallelizable training	✓
Fast autoregressive generation	X
Avoid vanishing gradients	✓

Matrix multiplications, modern hardware (GPUs/TPUs) is highly optimized for this (though quadratic complexity)

Again quadratic complexity, have to compare to all past keys and values each step
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O(1) maximum path length between tokens

Image Source: https://jalammar.github.io/illustrated-transformer/https://jalammar.github.io/illustrated-transformer/,
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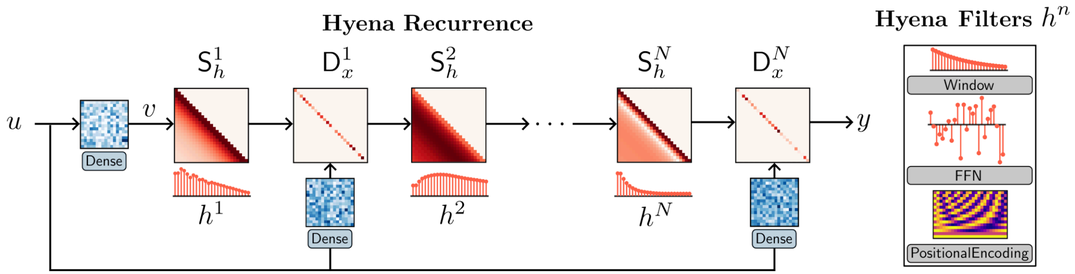
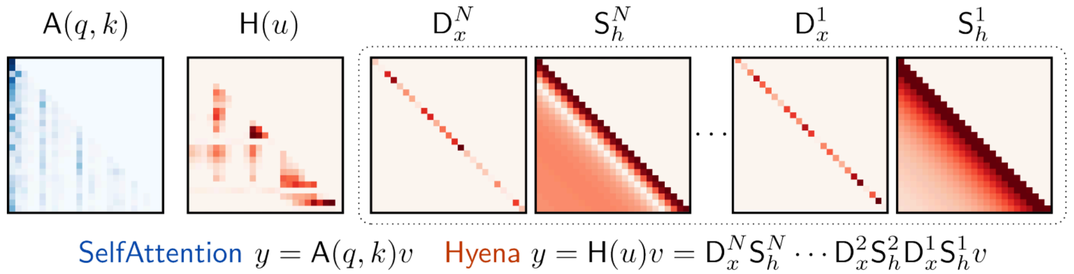
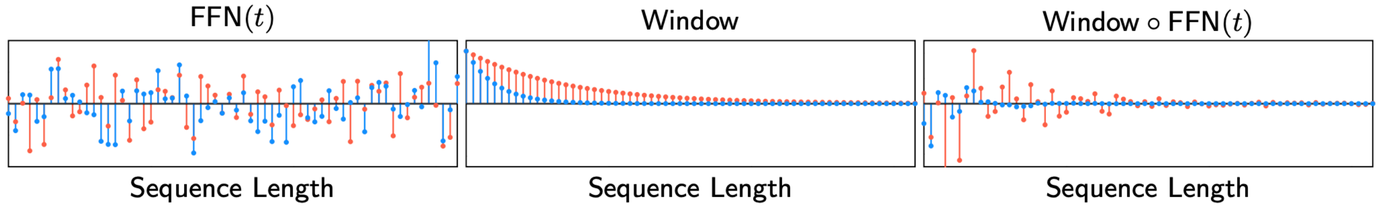
Long Convolutions







	Attention
Parallelizable training	✓
Fast autoregressive generation	X
Avoid vanishing gradients	✓

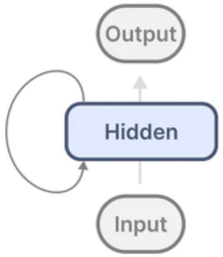
subquadratic complexity (FFTs)

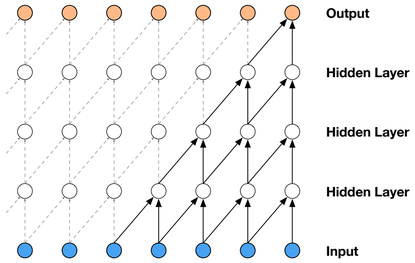
Quadratic complexity, but can distill into SSM post training (Laughing Hyena, Massaroli 2023)

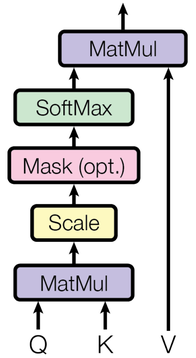
No recurrence to have to compute gradients through.

Poli et al. Hyena Hierarchy: Towards Larger Convolutional Language Models. 2023.
Massaroli et al. Laughing Hyena Distillery: Extracting Compact Recurrences From Convolutions. 2023.

Prior approaches to model long sequences







	Recurrent Neural Networks (RNNs)	Convolutions	Attention
Parallelizable training	X	✓	✓ **(Quadratic complexity)**
Fast autoregressive generation	✓	X	X
Avoid vanishing gradients	X	✓	✓

Attention Approximations

Linearized Attention

Attention

$$\boldsymbol{y}_i = \sum_{j=1}^{i} \frac{\exp(\boldsymbol{q}_i^\top \boldsymbol{k}_j / \sqrt{d}) \boldsymbol{v}_j}{\sum_{n=1}^{i} \exp(\boldsymbol{q}_i^\top \boldsymbol{k}_n / \sqrt{d})}$$

Linear Attention

$$\boldsymbol{y}_i = \sum_{j=1}^{i} \frac{\phi(\boldsymbol{q}_i)^\top \phi(\boldsymbol{k}_j) \boldsymbol{v}_j}{\sum_{n=1}^{i} \phi(\boldsymbol{q}_i)^\top \phi(\boldsymbol{k}_n)}$$

1. Outer product over key, value head dims
2. Sum over sequence length
3. Dot product over query, key-value head dims

3. 2. 1.

$$\boldsymbol{y}_i = \frac{\phi(\boldsymbol{q}_i)^\top \left(\sum_{j=1}^{i} \left(\phi(\boldsymbol{k}_j) \boldsymbol{v}_j^\top \right) \right)}{\phi(\boldsymbol{q}_i)^\top \sum_{n=1}^{i} \phi(\boldsymbol{k}_n)}$$

e.g. Linear Transformers (Katharopoulos 2020), Based (Arora 2024)

Image Source: https://hazyresearch.stanford.edu/blog/2023-12-11-zoology2-based, Child 2019: https://arxiv.org/abs/1904.10509v1

Attention Approximations

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e.g. Linear Transformers (Katharopoulos 2020), Based (Arora 2024)

Can also be viewed as a linear RNN at inference

Image Source: https://hazyresearch.stanford.edu/blog/2023-12-11-zoology2-based, Child 2019: https://arxiv.org/abs/1904.10509v1

Attention Approximations

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Linear Attention

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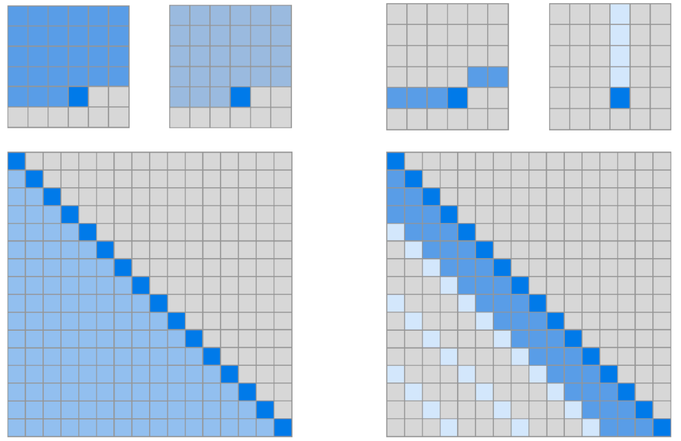
e.g. Linear Transformers (Katharopoulos 2020), Based (Arora 2024)

Sparse Attention



(a) Transformer (b) Sparse Transformer (strided)

e.g. Sparse Transformers (Child 2019), Big Bird (Zaheer 2020)

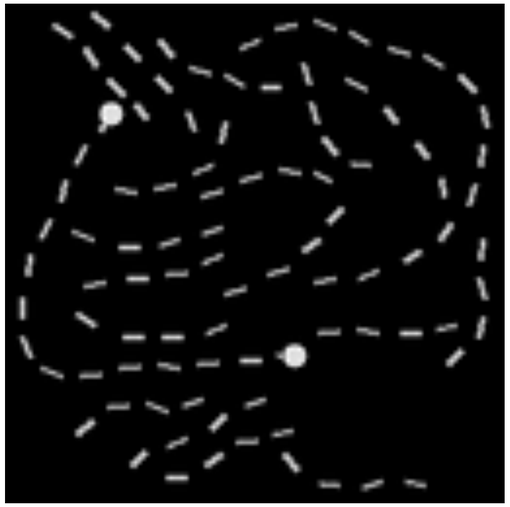
Image Source: https://hazyresearch.stanford.edu/blog/2023-12-11-zoology2-based, Child 2019: https://arxiv.org/abs/1904.10509v1

Long range benchmarks

Long Range Arena

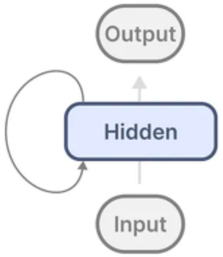
Model	ListOps	Text	Retrieval	Image	Pathfinder	Path-X	Avg
Random	10.00	50.00	50.00	10.00	50.00	50.00	36.67
Transformer	36.37	64.27	57.46	42.44	71.40	✗	53.66
Local Attention	15.82	52.98	53.39	41.46	66.63	✗	46.71
Sparse Trans.	17.07	63.58	59.59	44.24	71.71	✗	51.03
Longformer	35.63	62.85	56.89	42.22	69.71	✗	52.88
Linformer	35.70	53.94	52.27	38.56	76.34	✗	51.14
Reformer	<u>37.27</u>	56.10	53.40	38.07	68.50	✗	50.56
Sinkhorn Trans.	33.67	61.20	53.83	41.23	67.45	✗	51.23
Synthesizer	36.99	61.68	54.67	41.61	69.45	✗	52.40
BigBird	36.05	64.02	59.29	40.83	74.87	✗	54.17
Linear Trans.	16.13	<u>65.90</u>	53.09	42.34	75.30	✗	50.46
Performer	18.01	65.40	53.82	42.77	77.05	✗	51.18
FNet	35.33	65.11	59.61	38.67	<u>77.80</u>	✗	54.42
Nyströmformer	37.15	65.52	<u>79.56</u>	41.58	70.94	✗	57.46
Luna-256	37.25	64.57	79.29	<u>47.38</u>	77.72	✗	<u>59.37</u>

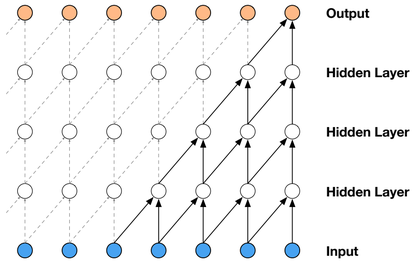
Path-X example:

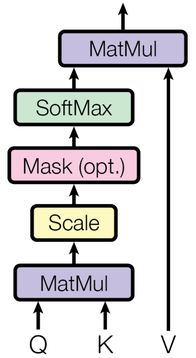


Tay et al. Long Range Arena: A Benchmark for Efficient Transformers. 2020.
Linsley et al. Learning long-range spatial dependencies with horizontal gated recurrent units. 2018.

Deep SSMs

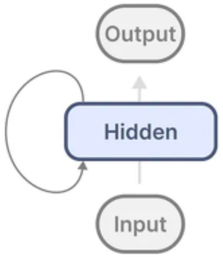


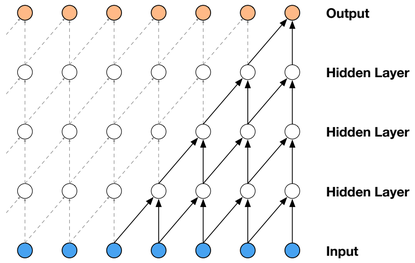


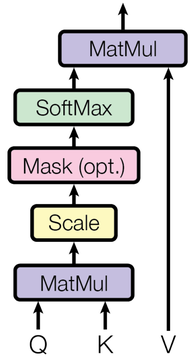


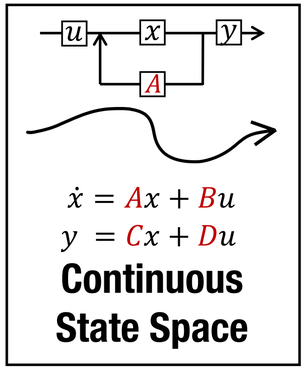
	Recurrent Neural Networks (RNNs)	Convolutions	Attention
Parallelizable training	X	✓	✓ **(Quadratic complexity)**
Fast autoregressive generation	✓	X	X
Avoid vanishing gradients	X	✓	✓

Deep SSMs









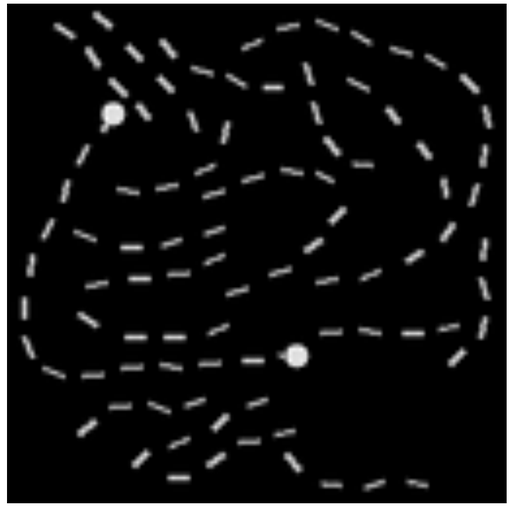
	Recurrent Neural Networks (RNNs)	Convolutions	Attention	Deep SSMs e.g. S4 (Gu et al. ICLR 2022)
Parallelizable training	X	✓	✓ (Quadratic complexity)	✓ (Subquadratic complexity)
Fast autoregressive generation	✓	X	X	✓
Avoid vanishing gradients	X	✓	✓	✓

S4 captures long-range dependencies

Long Range Arena

Model	ListOps	Text	Retrieval	Image	Pathfinder	Path-X	Avg
Random	10.00	50.00	50.00	10.00	50.00	50.00	36.67
Transformer	36.37	64.27	57.46	42.44	71.40	✗	53.66
Local Attention	15.82	52.98	53.39	41.46	66.63	✗	46.71
Sparse Trans.	17.07	63.58	59.59	44.24	71.71	✗	51.03
Longformer	35.63	62.85	56.89	42.22	69.71	✗	52.88
Linformer	35.70	53.94	52.27	38.56	76.34	✗	51.14
Reformer	37.27	56.10	53.40	38.07	68.50	✗	50.56
Sinkhorn Trans.	33.67	61.20	53.83	41.23	67.45	✗	51.23
Synthesizer	36.99	61.68	54.67	41.61	69.45	✗	52.40
BigBird	36.05	64.02	59.29	40.83	74.87	✗	54.17
Linear Trans.	16.13	65.90	53.09	42.34	75.30	✗	50.46
Performer	18.01	65.40	53.82	42.77	77.05	✗	51.18
FNet	35.33	65.11	59.61	38.67	77.80	✗	54.42
Nyströmformer	37.15	65.52	79.56	41.58	70.94	✗	57.46
Luna-256	37.25	64.57	79.29	47.38	77.72	✗	59.37
S4 (original)	58.35	76.02	87.09	87.26	86.05	88.10	80.48
S4 (updated)	**59.60**	**86.82**	**90.90**	**88.65**	**94.20**	**96.35**	**86.09**

Path-X example:



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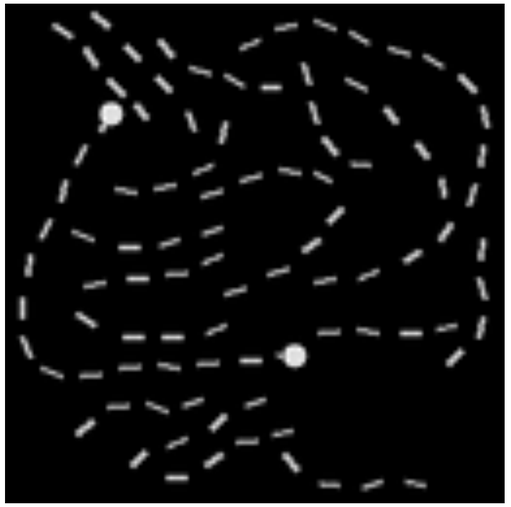
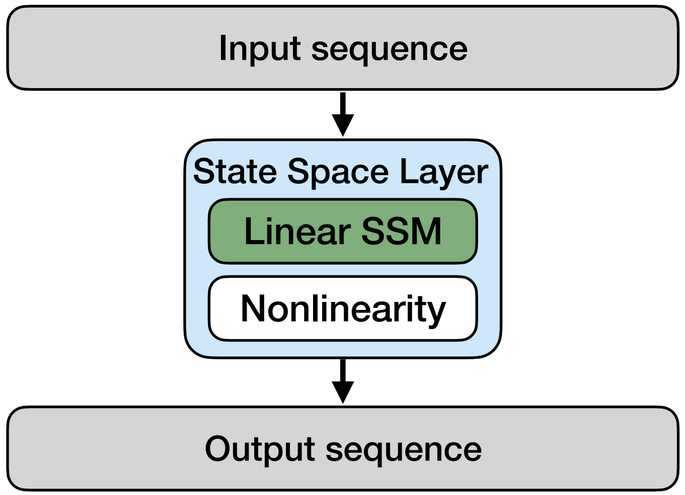


Agenda

- Introduction, motivation, prior approaches
- **Linear state space models (SSMs) overview**
- S4, convolutions, parameterization
- S5, diagonalization, parallel scans
- S6/Mamba/Hawk/Griffin, data-dependent dynamics
- Challenges and opportunities

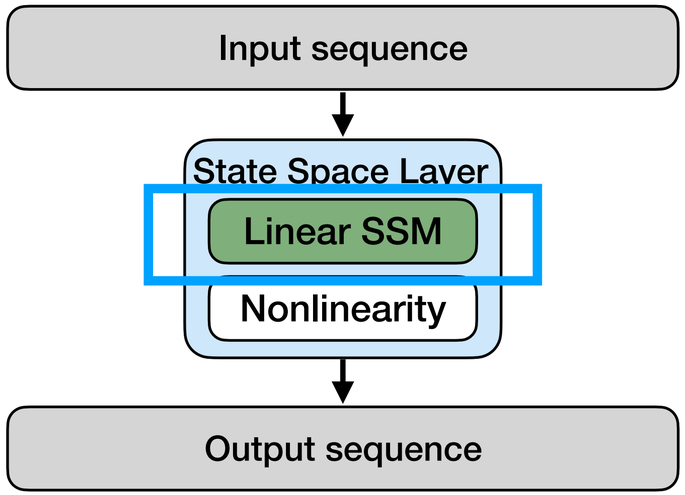
Key idea of Deep SSMs: Linear in time, nonlinear in depth

Key idea of Deep SSMs: Linear in time, nonlinear in depth



Key idea of Deep SSMs: Linear in time, nonlinear in depth

Continuous-time, linear state space model (SSM):



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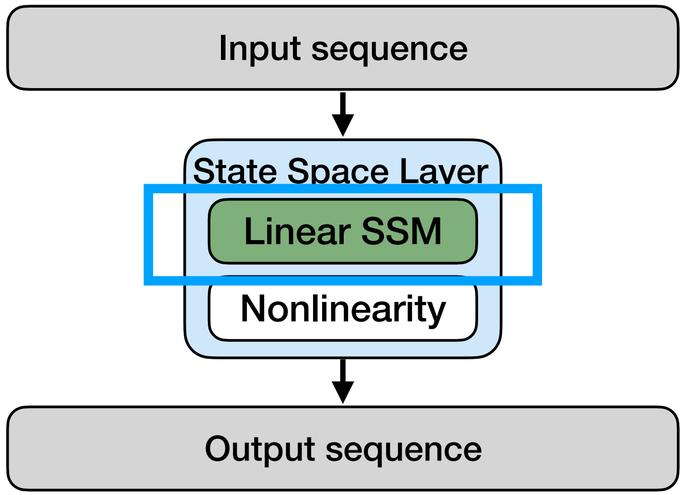

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Input Signal: $\mathbf{u}(t) \in \mathbb{R}^U$

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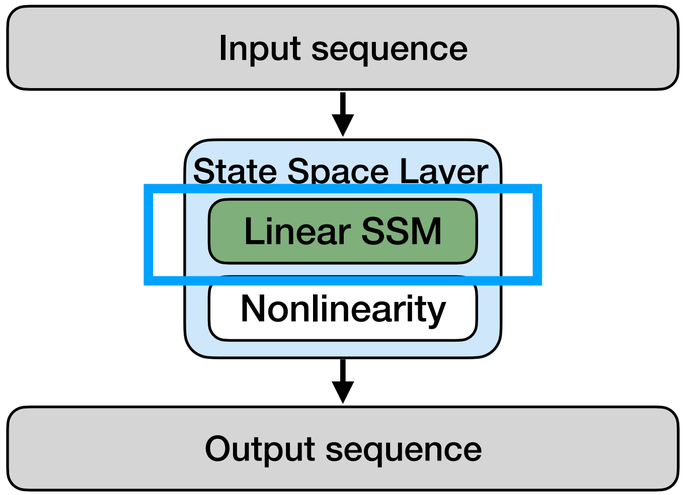
Key idea of Deep SSMs: Linear in time, nonlinear in depth



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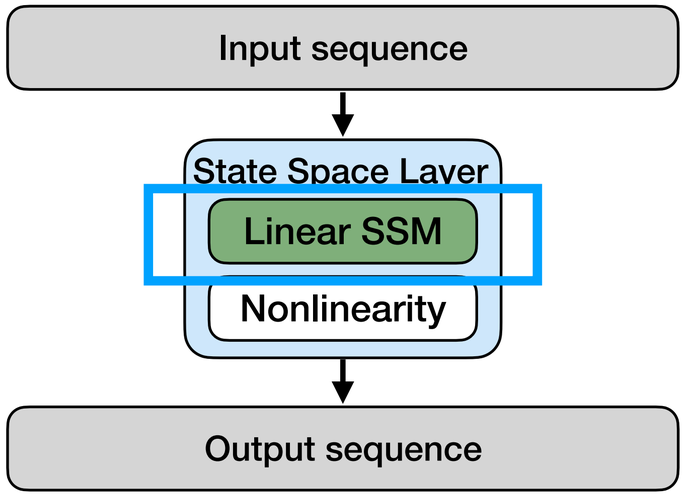
Key idea of Deep SSMs: Linear in time, nonlinear in depth



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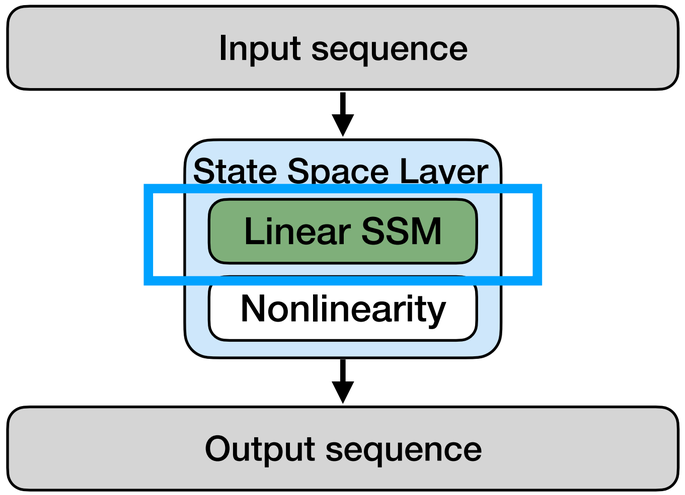

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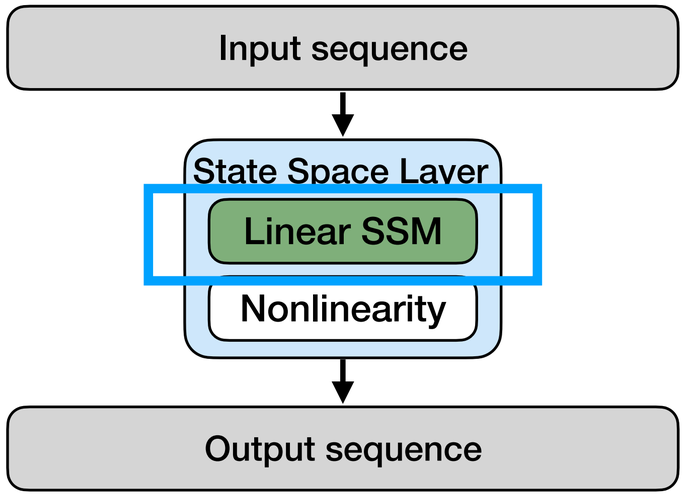

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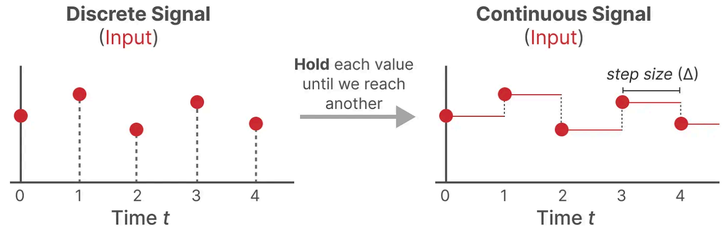
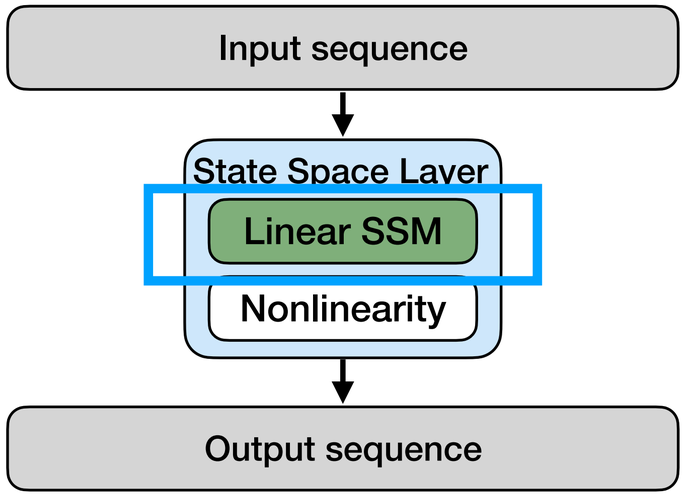


E.g. using Zero-order hold (ZOH):

$$\overline{\mathbf{A}} = e^{\mathbf{A}\Delta}, \quad \overline{\mathbf{B}} = \mathbf{A}^{-1}(\overline{\mathbf{A}} - \mathbf{I})\mathbf{B}, \quad \overline{\mathbf{C}} = \mathbf{C}, \quad \overline{\mathbf{D}} = \mathbf{D}$$

Key idea of Deep SSMs: Linear in time, nonlinear in depth



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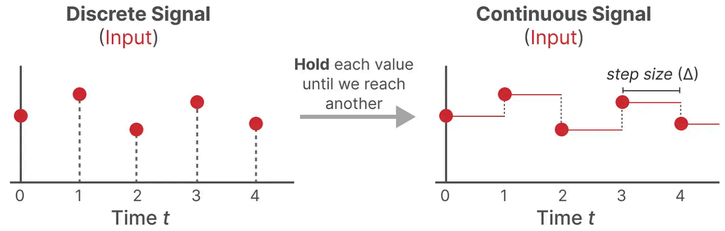

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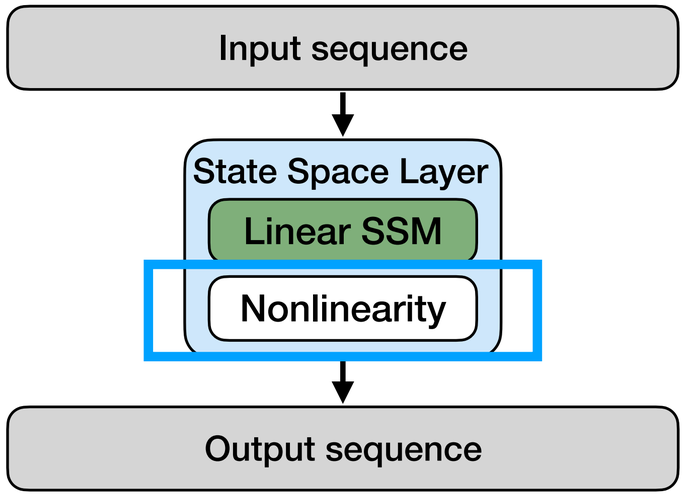
Treat Δ as learnable parameter.

Key idea of Deep SSMs: Linear in time, nonlinear in depth

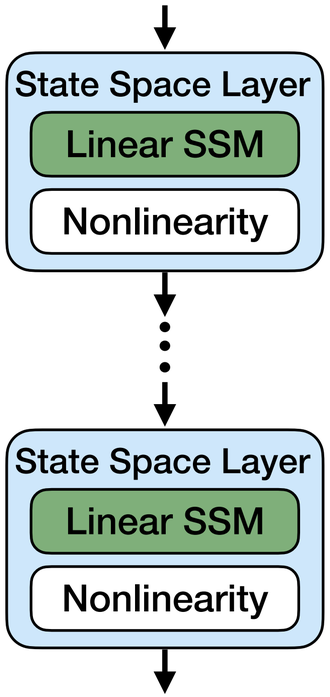


Nonlinear activation:

$$\mathbf{u}'_k = \mathbf{f}(\mathbf{y}_k)$$

E.g: gelu, GLU, layer norm, dropout, etc.

Key idea of Deep SSMs: Linear in time, nonlinear in depth



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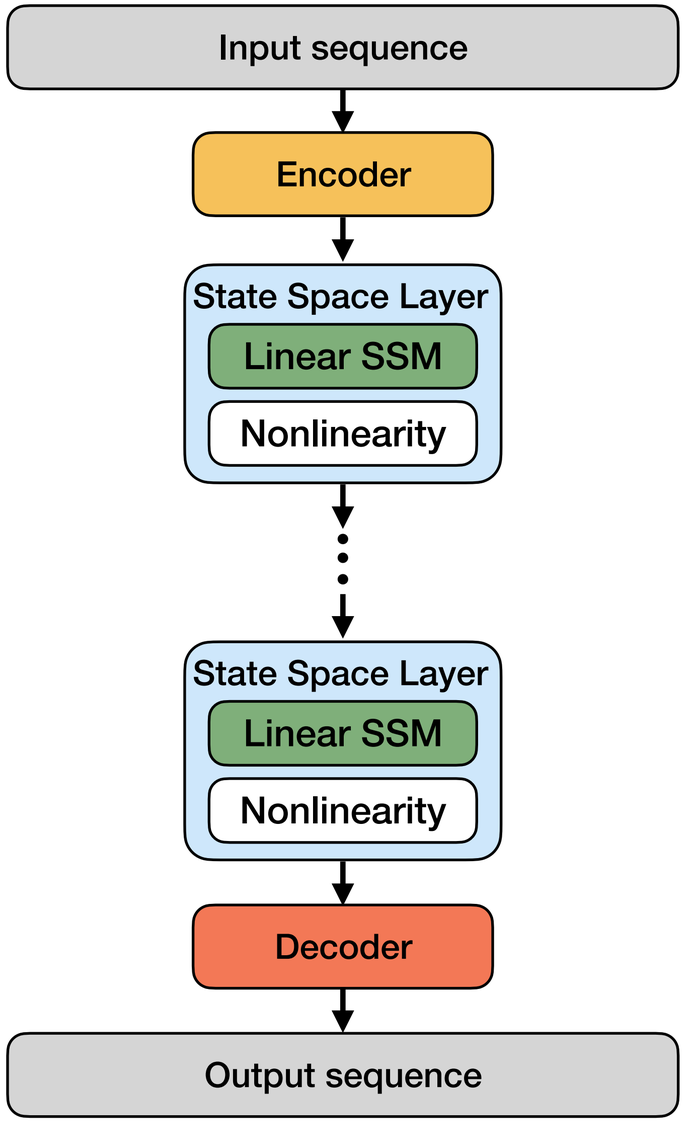

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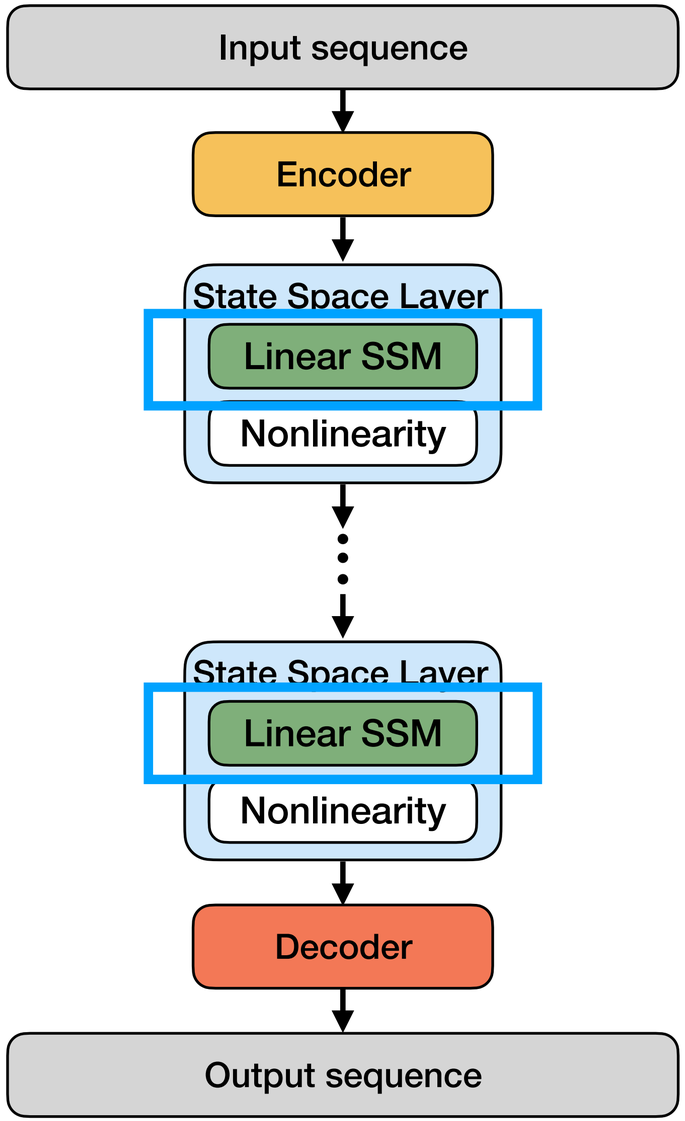
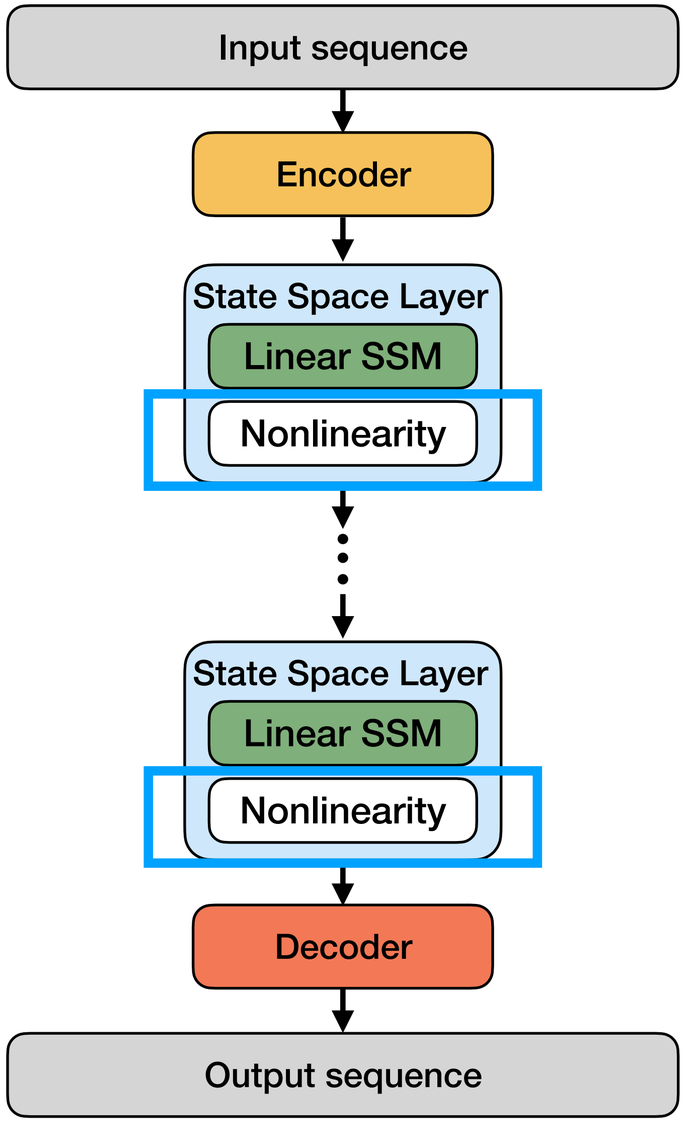


Linear in time: Efficient parallelization across the sequence

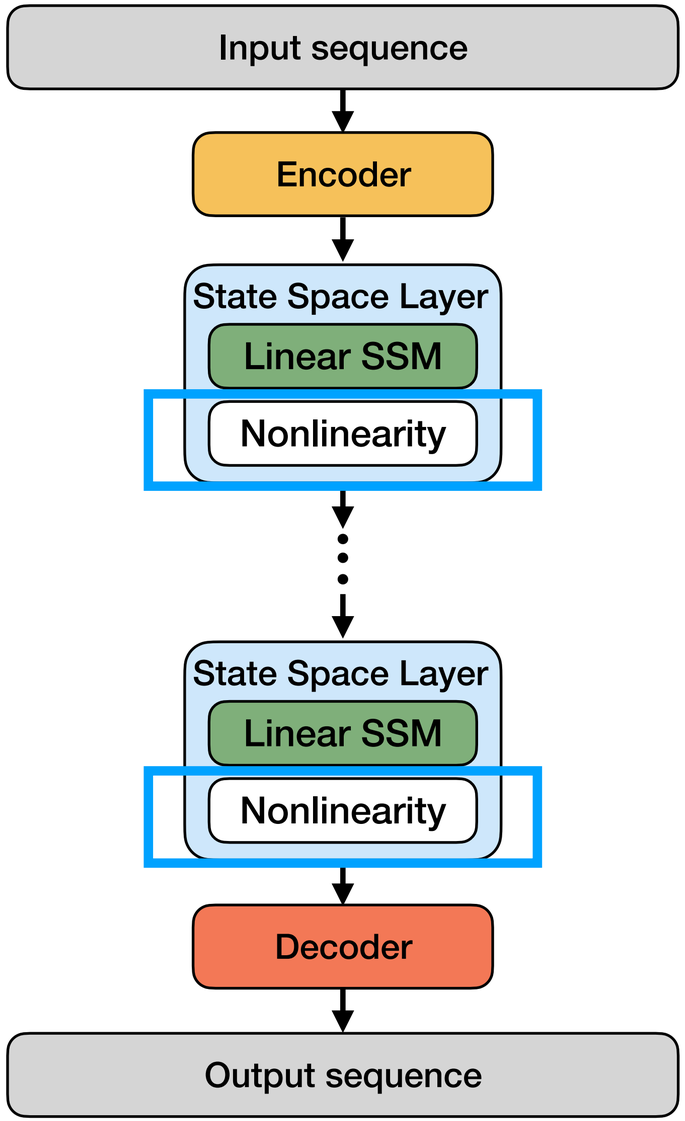
Key idea of Deep SSMs: Linear in time, nonlinear in depth



Linear in time: Efficient parallelization across the sequence

Nonlinear in depth: Stack of state space layers can represent nonlinear systems

Key idea of Deep SSMs: Linear in time, nonlinear in depth



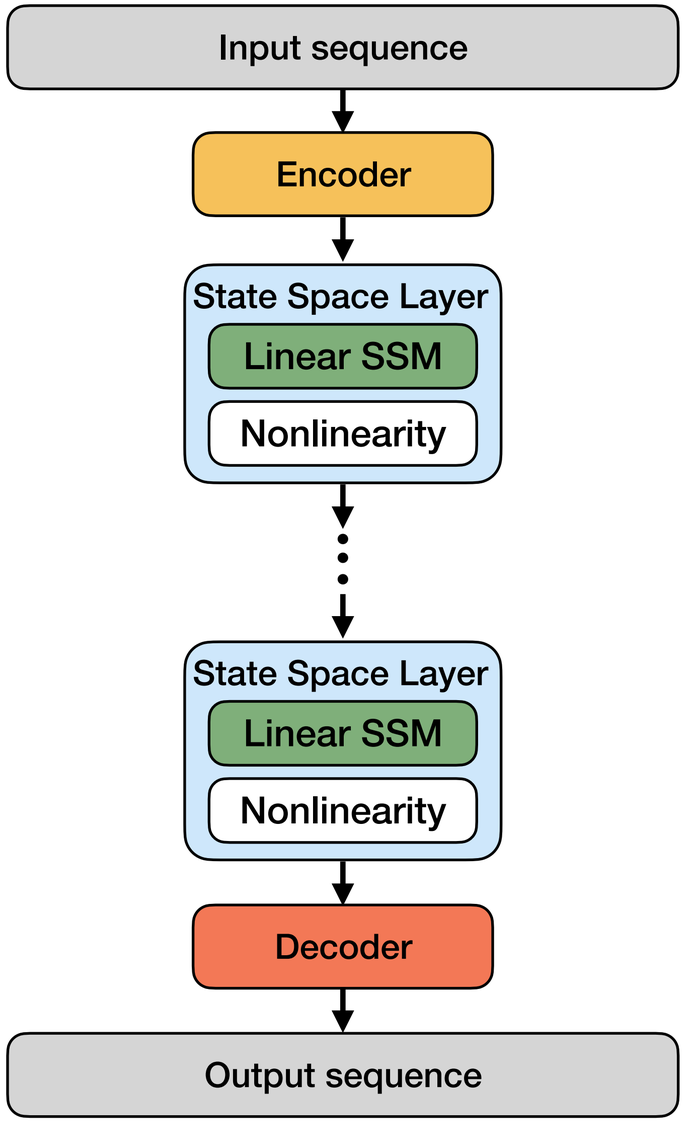
Linear in time: Efficient parallelization across the sequence

Nonlinear in depth: Stack of state space layers can represent nonlinear systems

Expressivity Results:

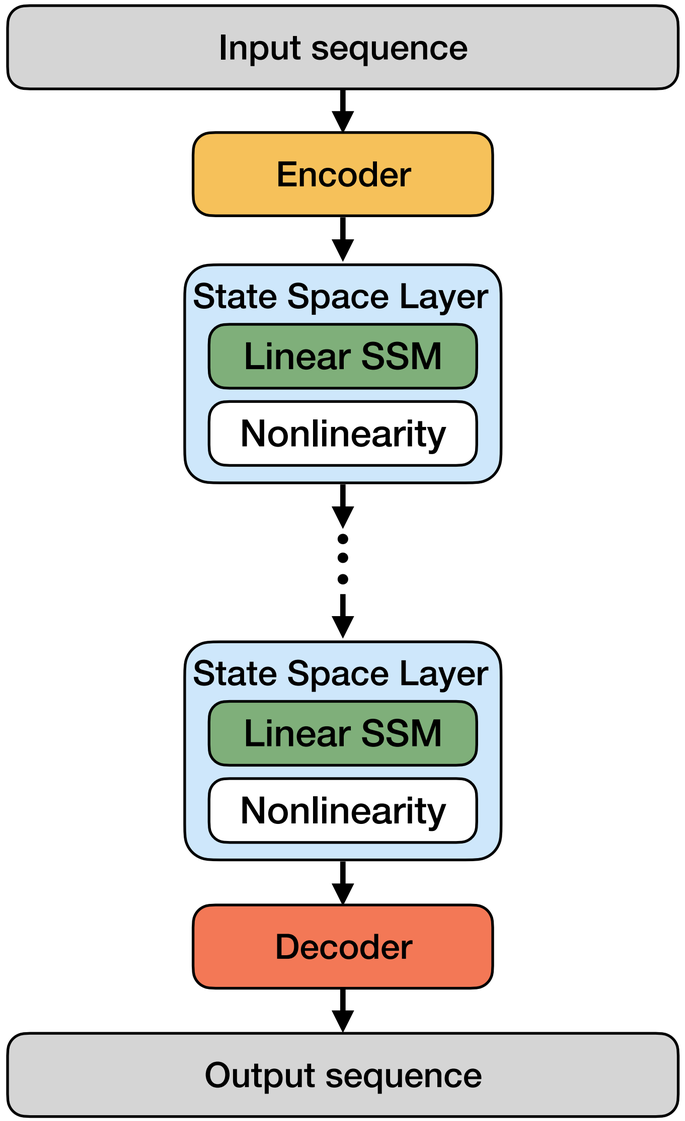
- Orvieto et al. 2023: https://arxiv.org/abs/2307.11888
- Wang et al. 2023: https://arxiv.org/abs/2309.13414

Key idea of Deep SSMs: Linear in time, nonlinear in depth



- Fast parallel processing
- Fast stateful autoregressive generation
- Can precisely parameterize to handle long-range dependencies (e.g. HiPPO framework, Gu et al. 2020)

Key idea of Deep SSMs: Linear in time, nonlinear in depth



Note, prior attempts at linear RNNs:

- QRNNs (Bradbury 2017)
- SRUs (Lei 2017)
- Linear surrogate RNNs (Martin 2018)

Likely reasons why more recent round of linear SSMs/RNNs have gained popularity:

- Improved parameterizations
- Ideas from Transformers, e.g. backbones, layer normalizations, etc.
- Improved parallel algorithm implementations

Agenda

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- Linear state space models (SSMs) overview
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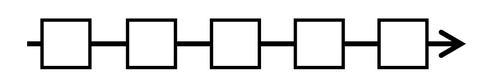
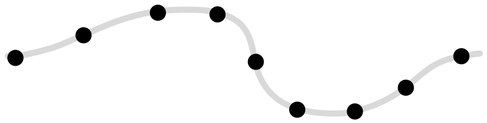
S4: Structured State Space Sequence Models

Gu et al. Efficiently modeling long sequences with structured state spaces. (2021)

S4 can be an RNN





$$x = \bar{A}x + \bar{B}u$$
$$y = \bar{C}x + \bar{D}u$$

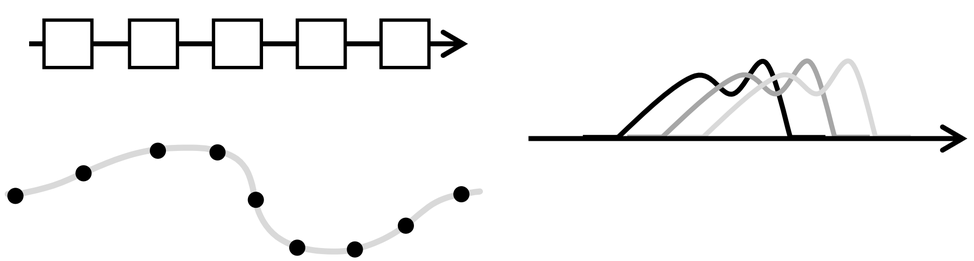
RNN

Fast Discrete Representations

S4 can be run as either an RNN for fast autoregressive generation

Gu et al. Efficiently modeling long sequences with structured state spaces. (2021)

S4 can be an RNN or a CNN



$$x = \bar{A}x + \bar{B}u$$
$$y = \bar{C}x + \bar{D}u$$

RNN

$$y = \overline{K} * u$$

CNN

Fast Discrete Representations

$$y_k = \overline{CA}^k \overline{B} u_0 + \overline{CA}^{k-1} \overline{B} u_1 + \cdots + \overline{CAB} u_{k-1} + \overline{CB} u_k$$
$$y = \overline{K} * u.$$

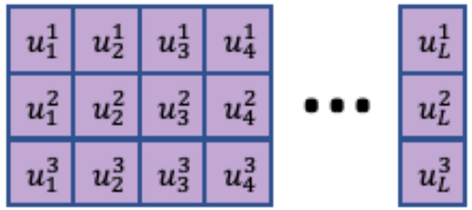
Convolution Kernel: $\overline{K} \in \mathbb{R}^L := (\overline{CB}, \overline{CAB}, \ldots, \overline{CA}^{L-1} \overline{B})$

S4 can be run as either an RNN or a CNN for fast parallel processing

Gu et al. Efficiently modeling long sequences with structured state spaces. (2021)

S4: Stack of single-input, single-output (SISO) SSMs

$u_{1:L} \in \mathbb{R}^{L \times 3}$



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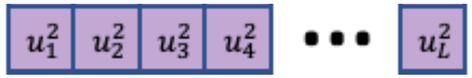

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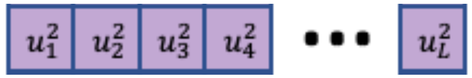

State Matrix: $\mathbf{A} \in \mathbb{R}^{N \times N}$

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$$\mathbf{x}_k = \overline{\mathbf{A}}\mathbf{x}_{k-1} + \overline{\mathbf{B}}\mathbf{u}_k$$

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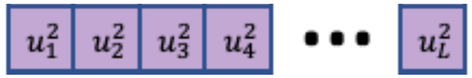

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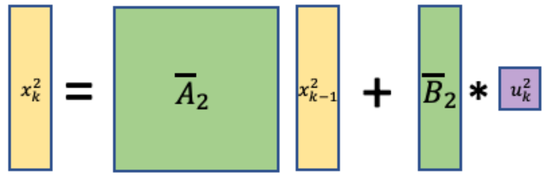

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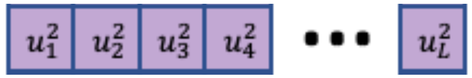

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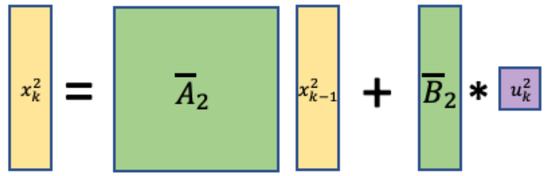

Output Matrix: $\mathbf{C} \in \mathbb{R}^{1 \times N}$

$$\mathbf{y}_k = \overline{\mathbf{C}}\mathbf{x}_k$$

S4: Stack of single-input, single-output (SISO) SSMs

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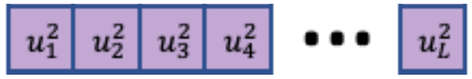

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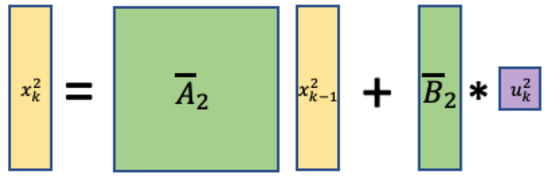

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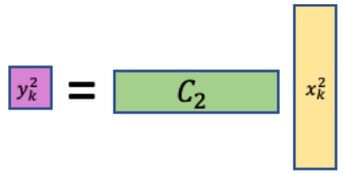

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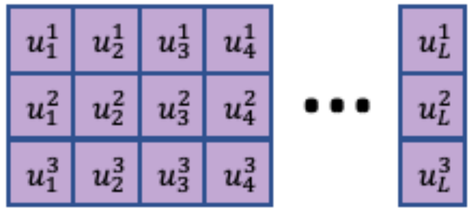

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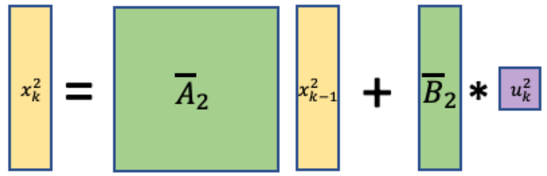

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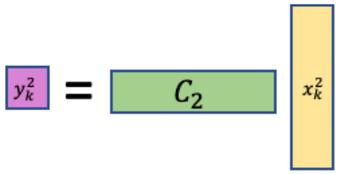

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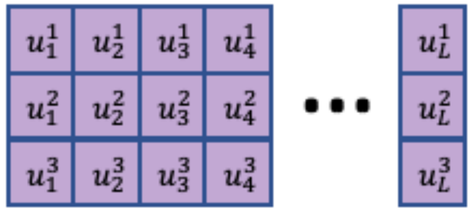

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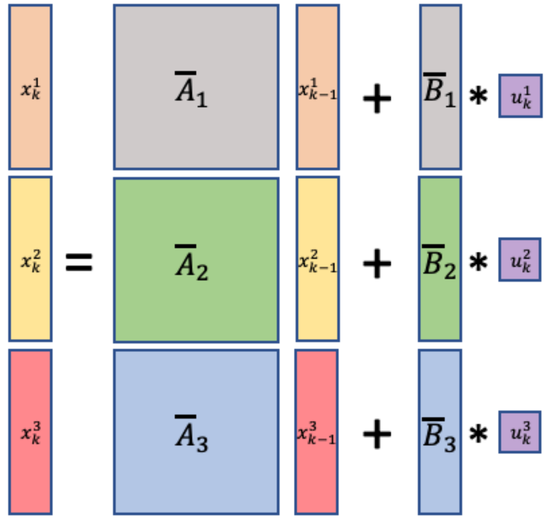

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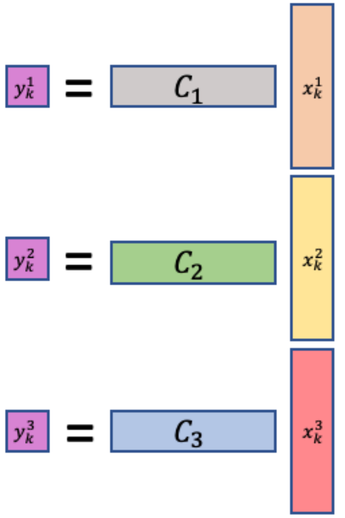

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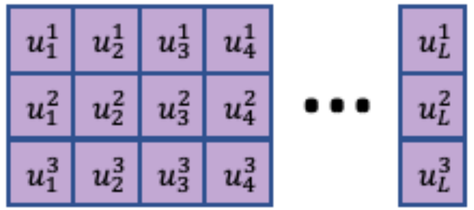

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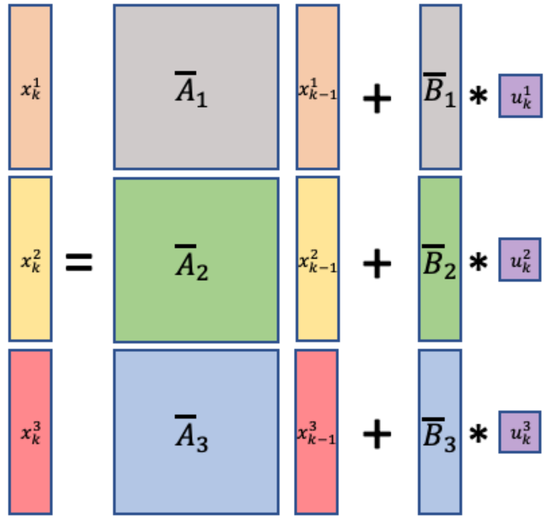

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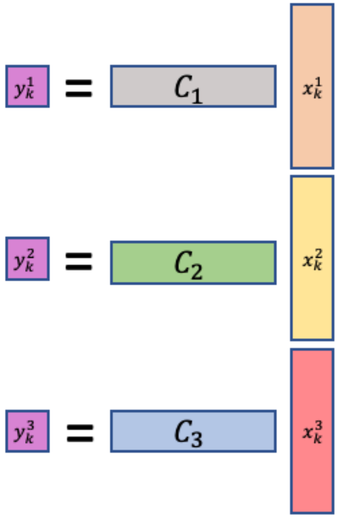
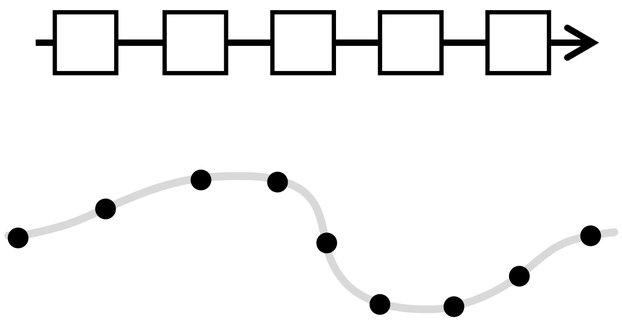


Motivation for this structure:

- **Computation (1D convolutions)**
- **Parameterization and Initialization (HiPPO, designed for SISO SSMs)**
- **Parameter efficient way to expand the state size**

S4 Computation: RNN Mode



$$\mathbf{x}_k = \overline{\mathbf{A}}\mathbf{x}_{k-1} + \overline{\mathbf{B}}\mathbf{u}_k$$

$$\mathbf{y}_k = \overline{\mathbf{C}}\mathbf{x}_k + \overline{\mathbf{D}}\mathbf{u}_k$$

S4 Computation: Convolution Mode

Consider a single S4 SSM:

$$\mathbf{x}_k = \overline{\mathbf{A}}\mathbf{x}_{k-1} + \overline{\mathbf{B}}\mathbf{u}_k$$

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Convolution equivalence holds for any linear time-invariant (LTI) system

Unroll the recurrence:

$$x_0 = \overline{\boldsymbol{B}}u_0$$

$$y_0 = \overline{\boldsymbol{CB}}u_0$$

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Convolution equivalence holds for any linear time-invariant (LTI) system

Unroll the recurrence:

$$x_0 = \overline{\boldsymbol{B}}u_0 \qquad x_1 = \overline{\boldsymbol{AB}}u_0 + \overline{\boldsymbol{B}}u_1$$

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Convolution equivalence holds for any linear time-invariant (LTI) system

Unroll the recurrence:

$$x_0 = \overline{\boldsymbol{B}}u_0 \qquad x_1 = \overline{\boldsymbol{AB}}u_0 + \overline{\boldsymbol{B}}u_1 \qquad x_2 = \overline{\boldsymbol{A}}^2\overline{\boldsymbol{B}}u_0 + \overline{\boldsymbol{AB}}u_1 + \overline{\boldsymbol{B}}u_2$$

$$y_0 = \overline{\boldsymbol{CB}}u_0 \qquad y_1 = \overline{\boldsymbol{CAB}}u_0 + \overline{\boldsymbol{CB}}u_1 \qquad y_2 = \overline{\boldsymbol{CA}}^2\overline{\boldsymbol{B}}u_0 + \overline{\boldsymbol{CAB}}u_1 + \overline{\boldsymbol{CB}}u_2$$

S4 Computation: Convolution Mode

Consider a single S4 SSM:

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$$y_0 = \overline{\boldsymbol{CB}}u_0 \qquad y_1 = \overline{\boldsymbol{CAB}}u_0 + \overline{\boldsymbol{CB}}u_1 \qquad y_2 = \overline{\boldsymbol{CA}}^2\overline{\boldsymbol{B}}u_0 + \overline{\boldsymbol{CAB}}u_1 + \overline{\boldsymbol{CB}}u_2$$

$$y_k = \overline{\boldsymbol{CA}}^k\overline{\boldsymbol{B}}u_0 + \overline{\boldsymbol{CA}}^{k-1}\overline{\boldsymbol{B}}u_1 + \cdots + \overline{\boldsymbol{CAB}}u_{k-1} + \overline{\boldsymbol{CB}}u_k$$

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Consider a single S4 SSM:

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Convolution equivalence holds for any linear time-invariant (LTI) system

Unroll the recurrence:

$$x_0 = \overline{\boldsymbol{B}}u_0 \qquad x_1 = \overline{\boldsymbol{AB}}u_0 + \overline{\boldsymbol{B}}u_1 \qquad x_2 = \overline{\boldsymbol{A}}^2\overline{\boldsymbol{B}}u_0 + \overline{\boldsymbol{AB}}u_1 + \overline{\boldsymbol{B}}u_2$$

$$y_0 = \overline{\boldsymbol{CB}}u_0 \qquad y_1 = \overline{\boldsymbol{CAB}}u_0 + \overline{\boldsymbol{CB}}u_1 \qquad y_2 = \overline{\boldsymbol{CA}}^2\overline{\boldsymbol{B}}u_0 + \overline{\boldsymbol{CAB}}u_1 + \overline{\boldsymbol{CB}}u_2$$

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$$y = \overline{\boldsymbol{K}} * u.$$

S4 Computation: Convolution Mode

Consider a single S4 SSM:

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Unroll the recurrence:

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Convolution kernel: $\overline{\boldsymbol{K}} \in \mathbb{R}^L := (\overline{\boldsymbol{CB}}, \overline{\boldsymbol{CAB}}, \ldots, \overline{\boldsymbol{CA}}^{L-1}\overline{\boldsymbol{B}})$

S4 Computation: Convolution Mode

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$$y_k = \overline{\boldsymbol{C}}\,\overline{\boldsymbol{A}}^k\overline{\boldsymbol{B}}u_0 + \overline{\boldsymbol{C}}\,\overline{\boldsymbol{A}}^{k-1}\overline{\boldsymbol{B}}u_1 + \cdots + \overline{\boldsymbol{C}\boldsymbol{A}\boldsymbol{B}}u_{k-1} + \overline{\boldsymbol{C}\boldsymbol{B}}u_k$$

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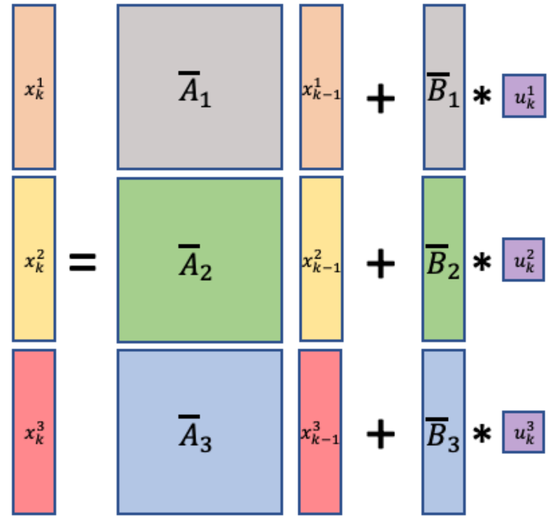
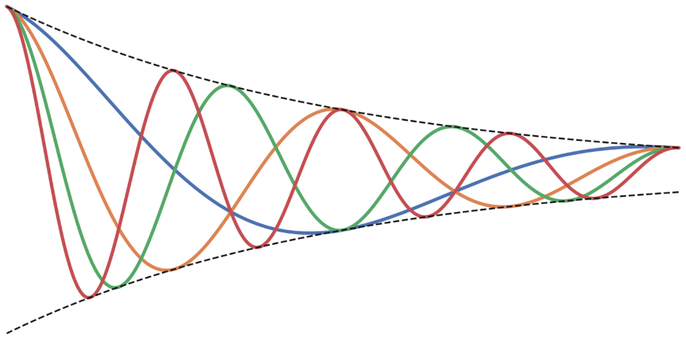
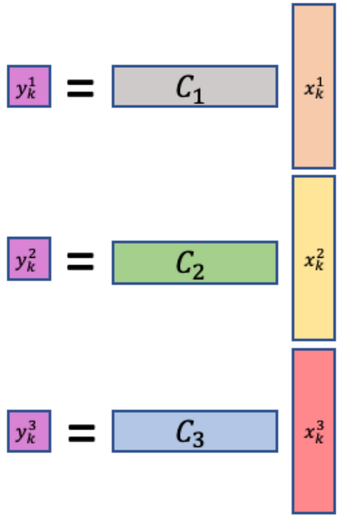


Stack of SISO SSMs gives a range of 1D convolution kernels that that can capture different timescales:





S4 Computation: Convolution Mode

Consider a single S4 SSM:

$$\mathbf{x}_k = \overline{\mathbf{A}}\mathbf{x}_{k-1} + \overline{\mathbf{B}}\mathbf{u}_k$$

$$\mathbf{y}_k = \overline{\mathbf{C}}\mathbf{x}_k + \overline{\mathbf{D}}\mathbf{u}_k$$

$$y_k = \overline{\boldsymbol{C}}\,\overline{\boldsymbol{A}}^k\overline{\boldsymbol{B}}u_0 + \overline{\boldsymbol{C}}\,\overline{\boldsymbol{A}}^{k-1}\overline{\boldsymbol{B}}u_1 + \cdots + \overline{\boldsymbol{C}\boldsymbol{A}\boldsymbol{B}}u_{k-1} + \overline{\boldsymbol{C}\boldsymbol{B}}u_k$$

$$y = \overline{\boldsymbol{K}} * u.$$

Convolution kernel: $\overline{\boldsymbol{K}} \in \mathbb{R}^L := (\overline{\boldsymbol{C}\boldsymbol{B}}, \overline{\boldsymbol{C}\boldsymbol{A}\boldsymbol{B}}, \ldots, \overline{\boldsymbol{C}}\,\overline{\boldsymbol{A}}^{L-1}\overline{\boldsymbol{B}})$

Convolution Theorem:

$$\mathcal{F}[f * g] = \mathcal{F}[f]\,\mathcal{F}[g].$$

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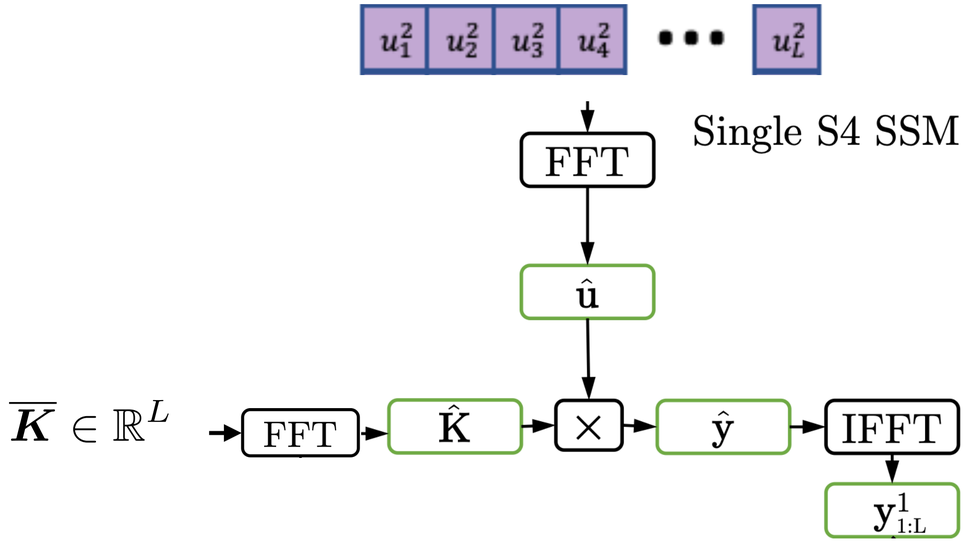

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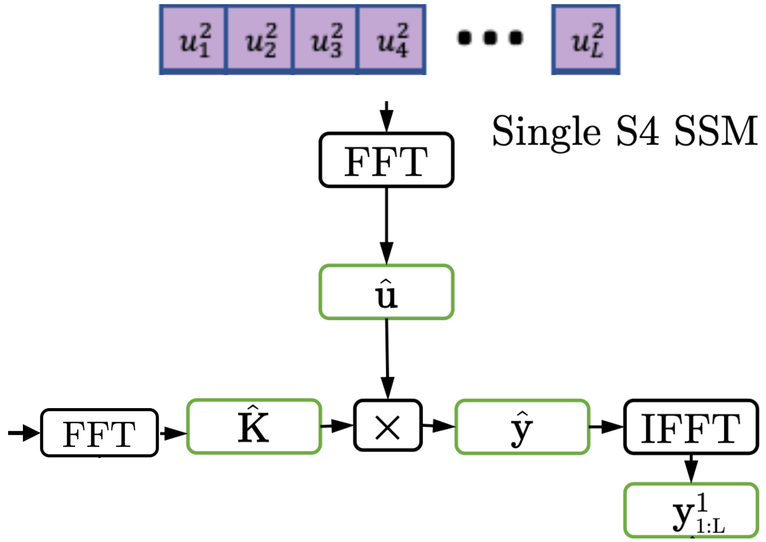
Convolution Theorem:

$$\mathcal{F}[f * g] = \mathcal{F}[f]\,\mathcal{F}[g].$$

Given kernel, the convolution can be computed with O(L logL) cost and O(L) space

Importantly, can be parallelized across the sequence

$\overline{\boldsymbol{K}} \in \mathbb{R}^L$



S4 Computation: Convolution Mode

Consider a single S4 SSM:

$$y_k = \overline{\boldsymbol{C}}\,\overline{\boldsymbol{A}}^k\overline{\boldsymbol{B}}u_0 + \overline{\boldsymbol{C}}\,\overline{\boldsymbol{A}}^{k-1}\overline{\boldsymbol{B}}u_1 + \cdots + \overline{\boldsymbol{C}}\,\overline{\boldsymbol{A}}\overline{\boldsymbol{B}}u_{k-1} + \overline{\boldsymbol{C}}\overline{\boldsymbol{B}}u_k$$

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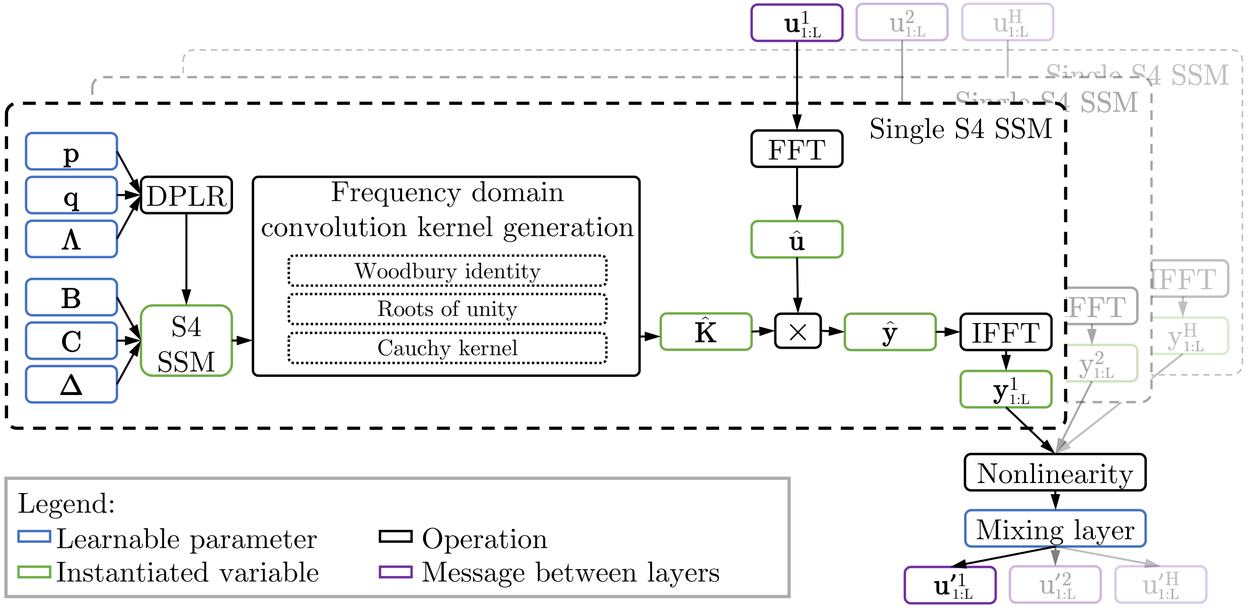
Total cost of S4 layer: $\mathcal{O}(H^2L + HL\log L)$

Nonlinear FFN Convolutions (H channels)

**(If we can compute kernel efficiently….
but this requires successive powers of A…)**

- **Naively, computing the kernel requires O(N^2L) operations**
- **If dynamics matrix is diagonal, Vandermonde matrices can be used, O(NL) time and space naively, but can in theory be cheaper**
- **S4 used a diagonal plus low rank (DPLR) dynamics matrix, so required a sophisticated algorithm which resulted in the use of Cauchy kernels**



S4 Parameterization and Initialization

Main idea: Using HiPPO Theory (Gu et al. 2020), can represent history of a scalar signal using a SISO linear SSM with special state matrix, HiPPO matrices.

Gu et al. HiPPO: Recurrent Memory with Optimal Polynomial Projections. (2020)

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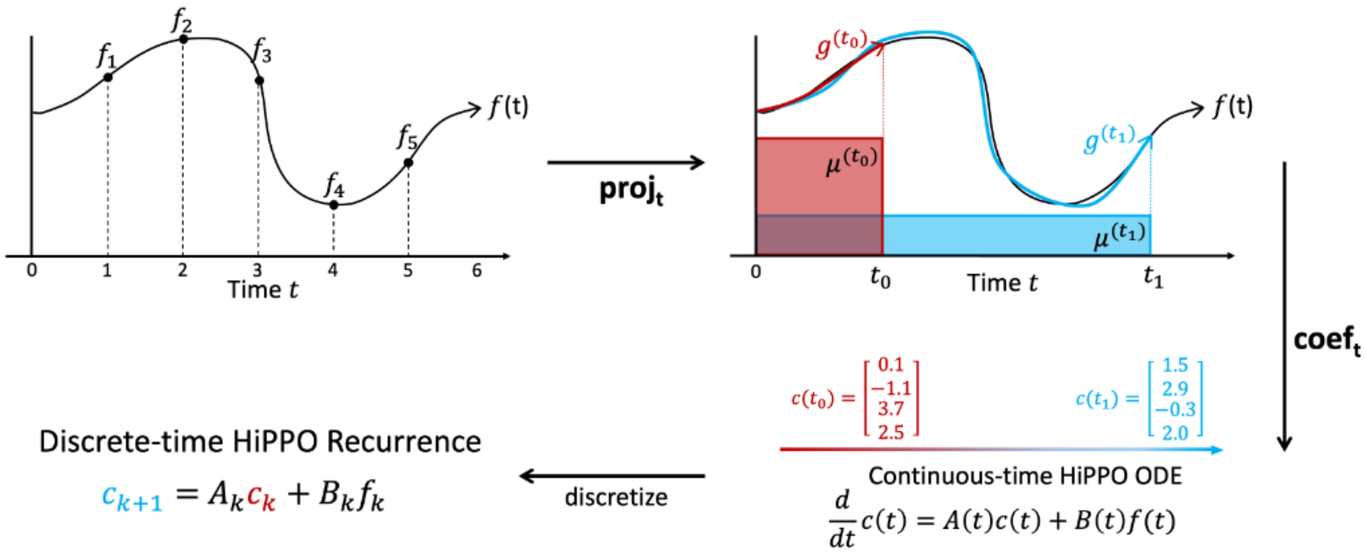

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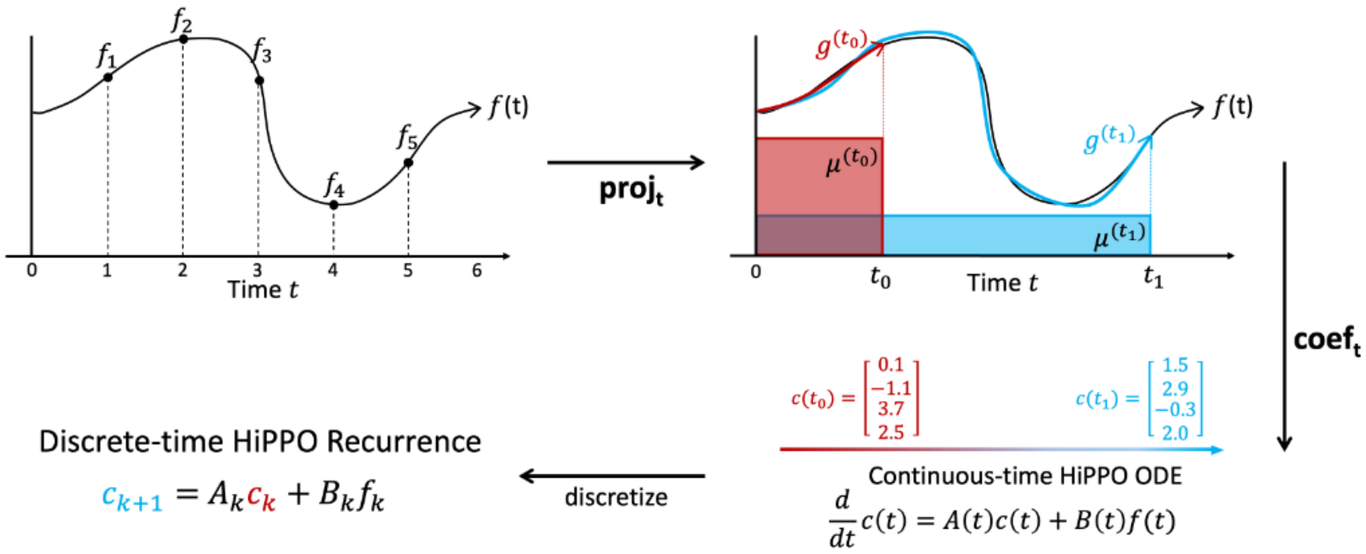

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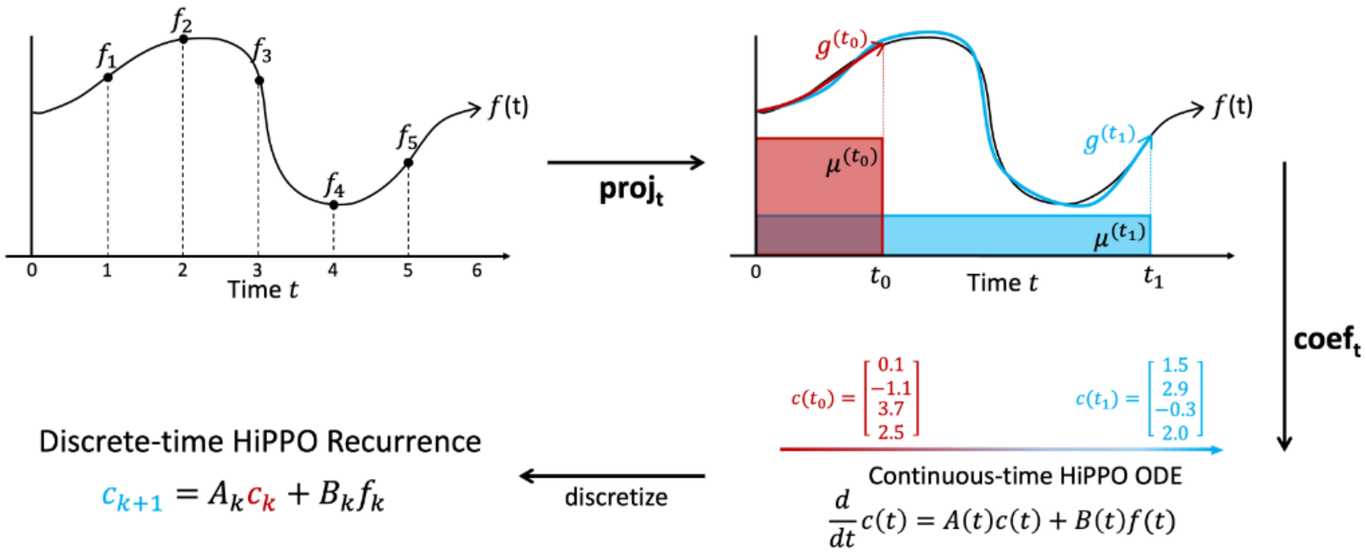

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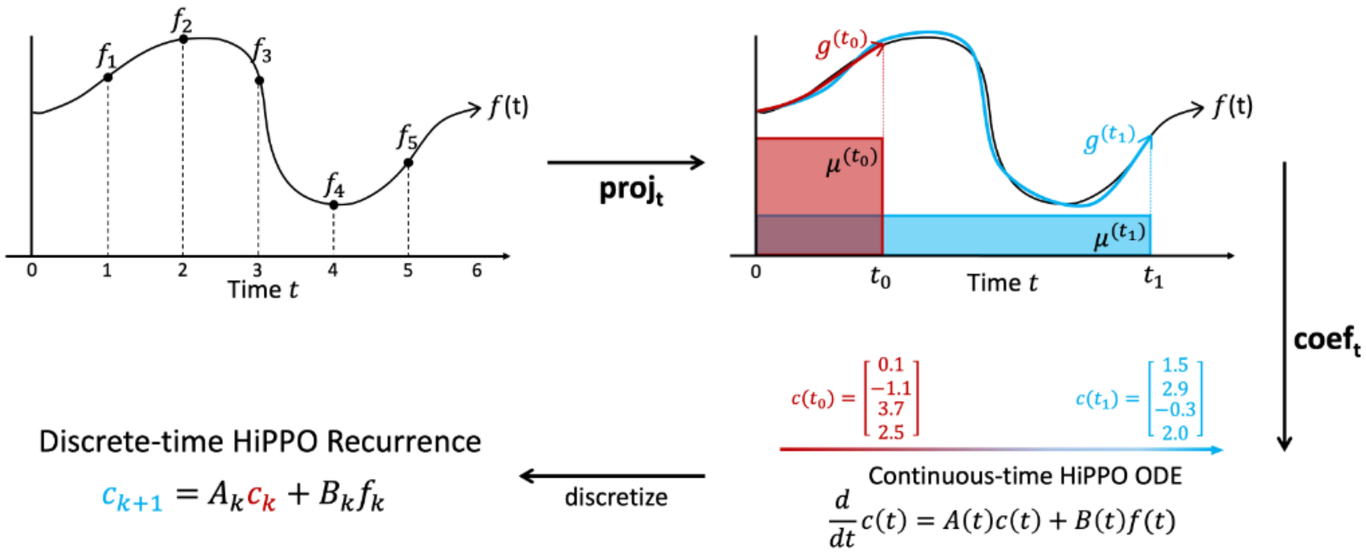

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From S4 paper:

Theorem 1. *All HiPPO matrices from [16] have a NPLR representation*

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^* - \boldsymbol{P}\boldsymbol{Q}^\top = \boldsymbol{V}\left(\boldsymbol{\Lambda} - (\boldsymbol{V}^*\boldsymbol{P})(\boldsymbol{V}^*\boldsymbol{Q})^*\right)\boldsymbol{V}^* \tag{6}$$

for unitary $\boldsymbol{V} \in \mathbb{C}^{N\times N}$*, diagonal* $\boldsymbol{\Lambda}$*, and low-rank factorization* $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{N\times r}$*. These matrices HiPPO- LegS, LegT, LagT all satisfy* $r = 1$ *or* $r = 2$*. In particular, equation* (2) *is NPLR with* $r = 1$*.*

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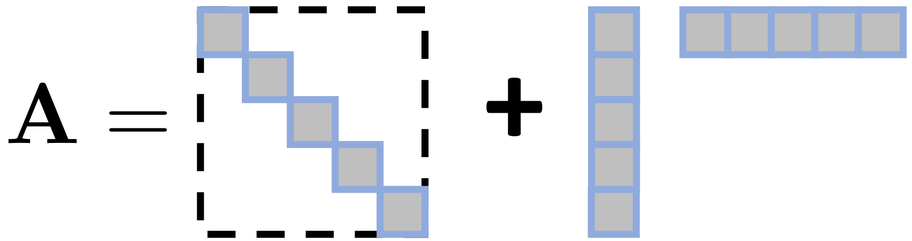

We will discuss what these matrices are doing more later.

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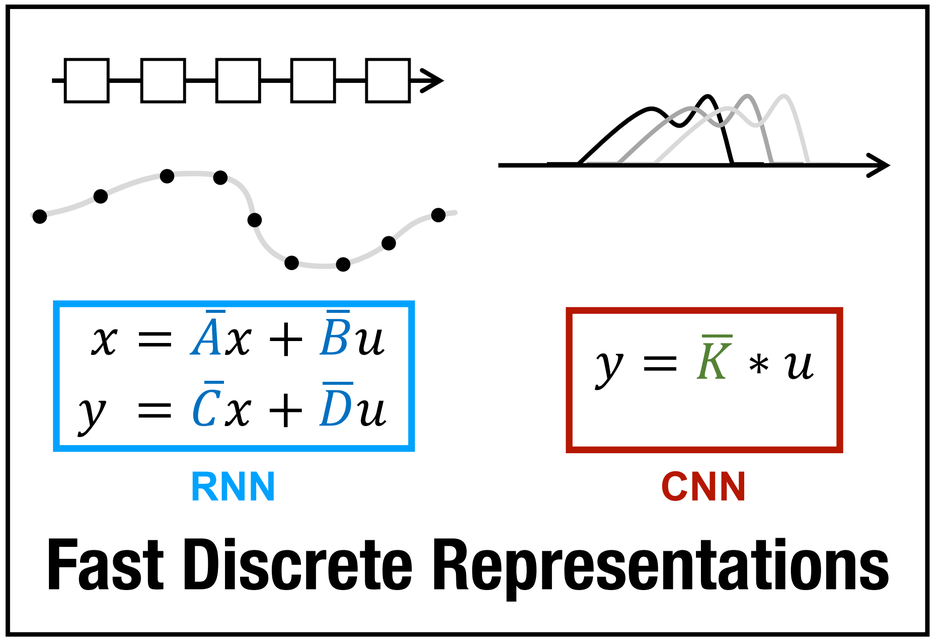
Background: S4 needs to be an RNN and CNN



S4 needs to be both an RNN and a CNN

This is elegant! But there are limitations:

- **CNN mode requires time-invariant system**
- **CNN mode cannot easily access states**
- **Complicated implementation**

Agenda

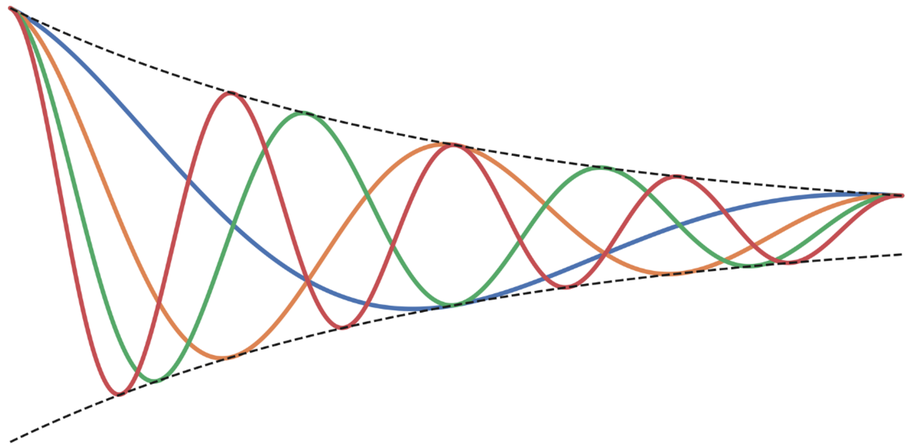
- Introduction, motivation, prior approaches
- Linear state space models (SSMs) overview
- S4, convolutions, parameterization
- **S5, diagonalization, parallel scans**
- S6/Mamba/Hawk/Griffin, data-dependent dynamics
- Challenges and opportunities

Can we get the same parallelizability, efficiency and performance, as S4 while addressing these limitations?

Smith, Warrington, Linderman. Simplified State Space Layers for Sequence Modeling. 2022.

From S4 to S5: Fully Recurrent

Convolution



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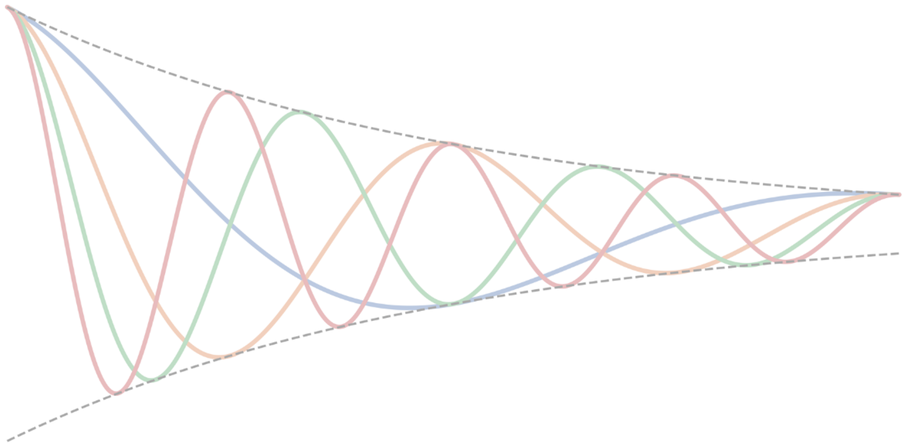
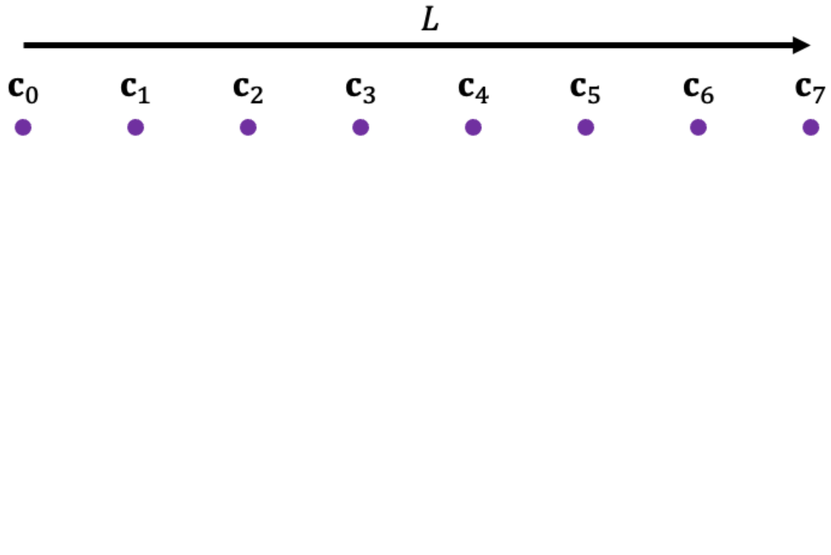


Parallel scan (prefix-sum)



Convolution limitations:

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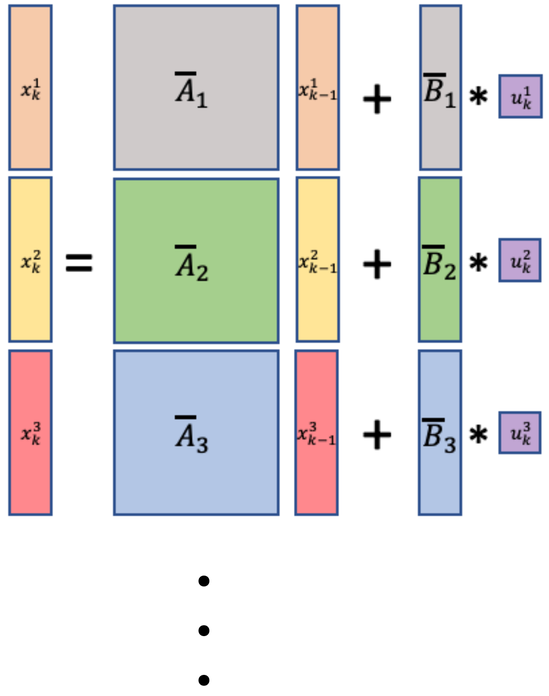
Scan allows:

- **Time-varying systems**
- **Access to states (parallel or autoregressive)**

From S4 to S5:

From S4 to S5: SISO to MIMO

Independent single-input, single-output (SISO) sequence maps



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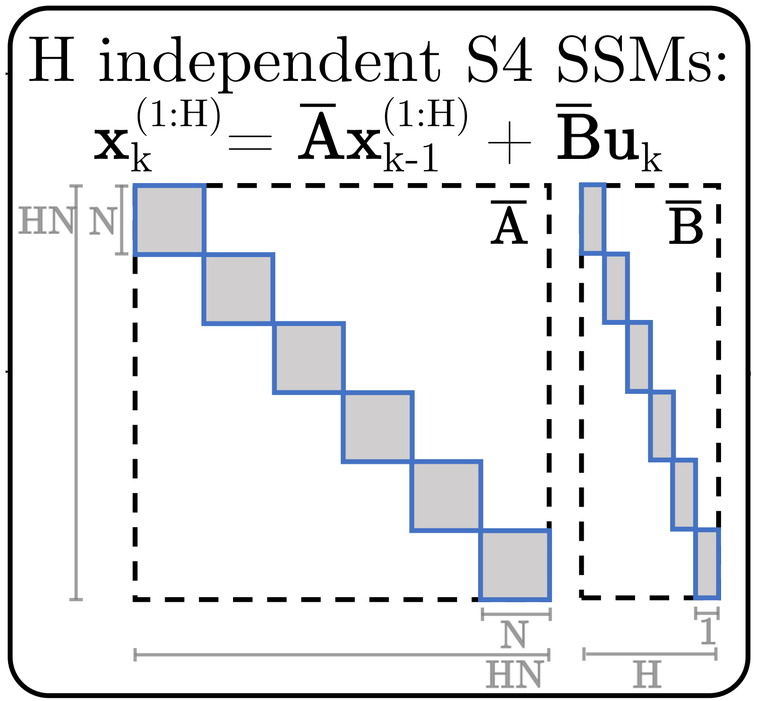

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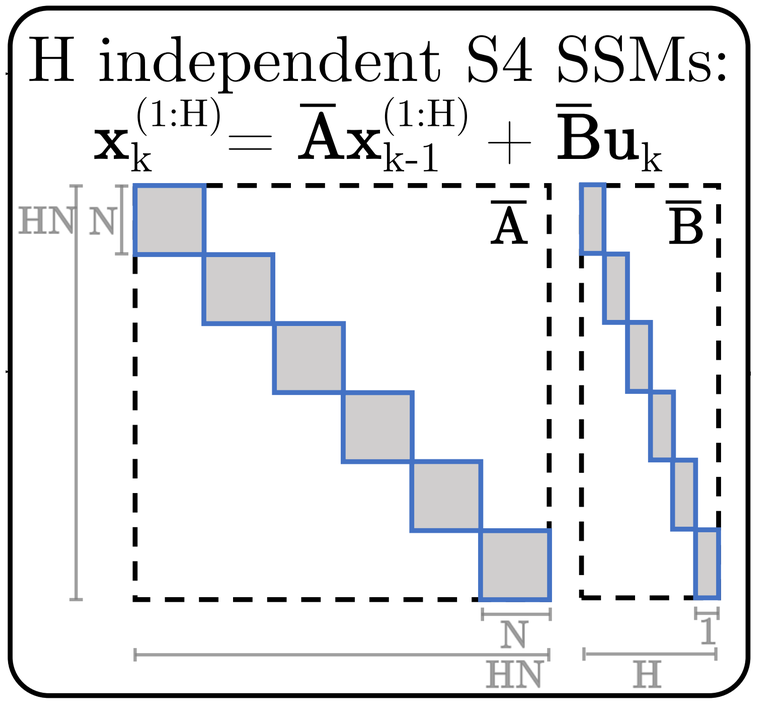

Large effective state size prevents the use of (basic) parallel scans.

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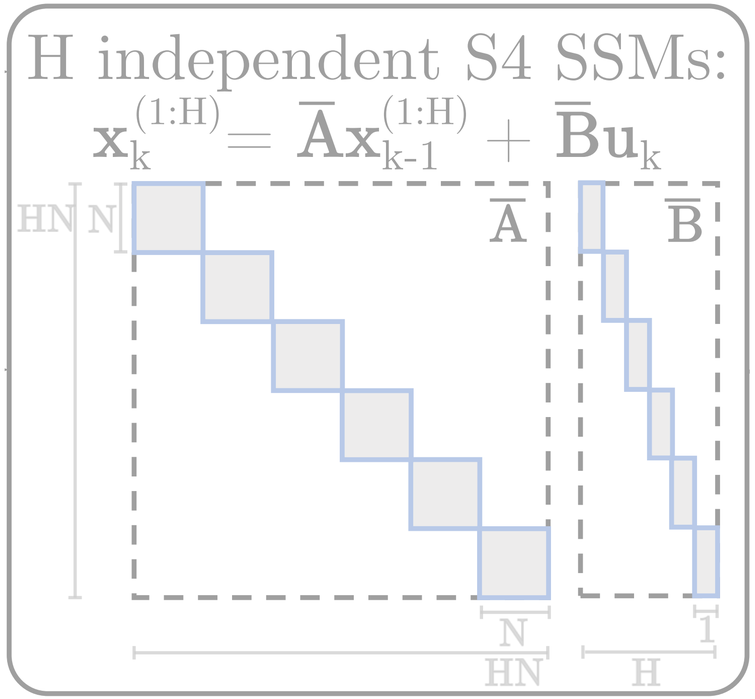

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One
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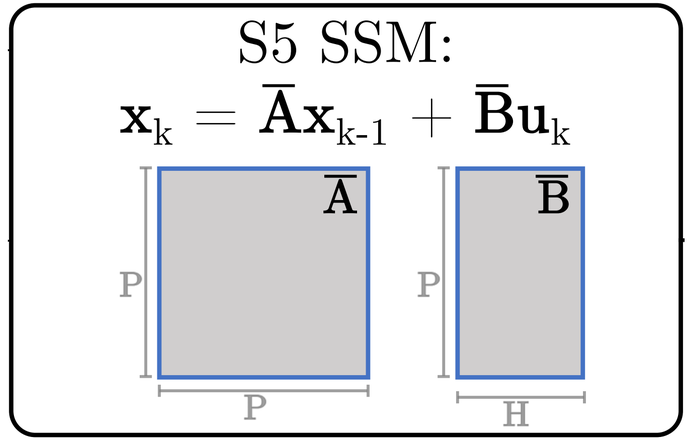

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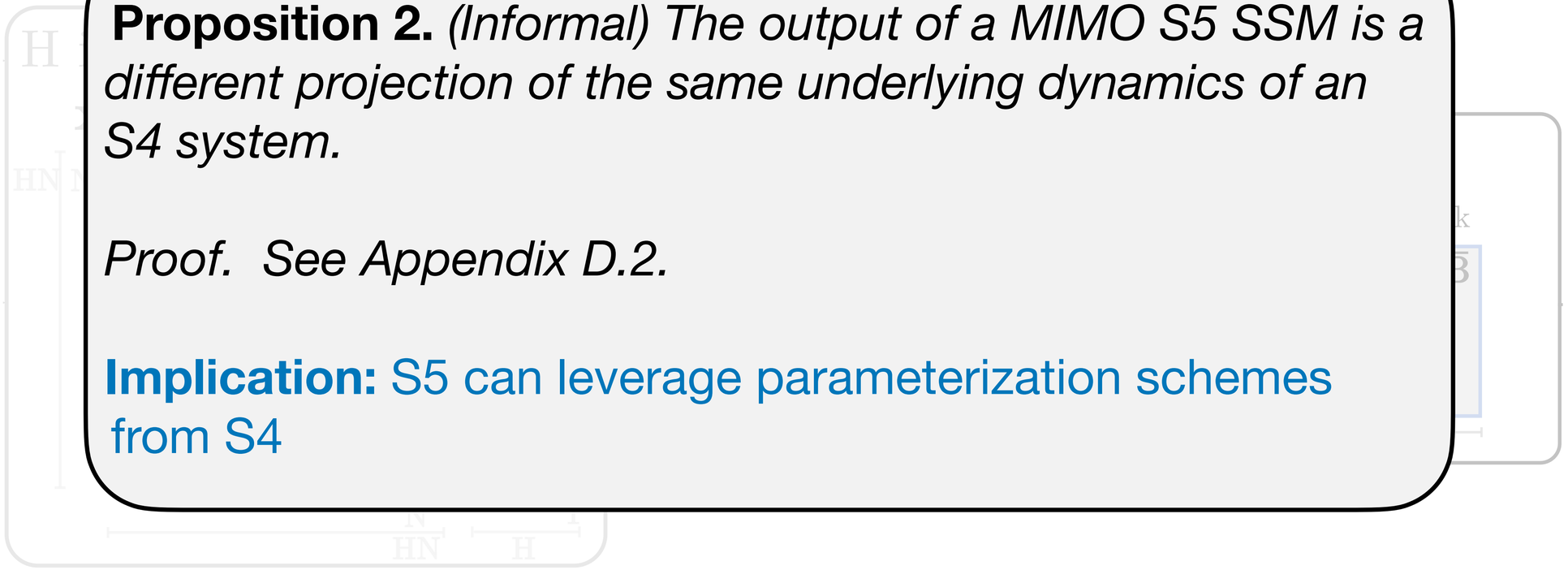

Proposition 2. *(Informal) The output of a MIMO S5 SSM is a different projection of the same underlying dynamics of an S4 system.*

Proof. See Appendix D.2.

Implication: S5 can leverage parameterization schemes from S4

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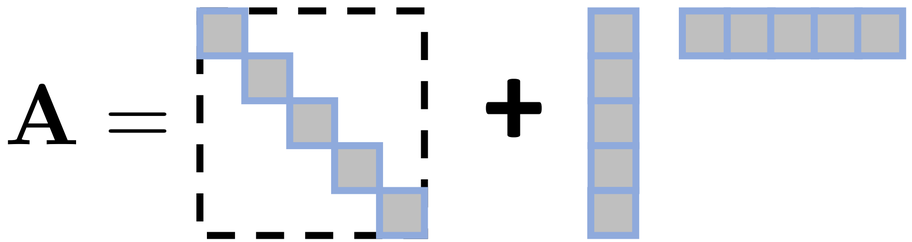
From S4 to S5: Diagonalized dynamics

Diagonal plus low-rank state matrix



From S4 to S5: Diagonalized dynamics

Diagonal plus low-rank
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Similar to findings from DSS (Gupta et al. 2022)
and S4D (Gu et al. 2022)

Importance of parameterization (e.g. HiPPO)?

$$\mathbf{x}_k = \overline{\mathbf{A}}\mathbf{x}_{k-1} + \overline{\mathbf{B}}\mathbf{u}_k$$

Image source: Orvieto et al. Resurrecting Recurrent Neural Networks for Long Sequences. 2023

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Note: real-valued diagonal matrices would be restricted in expressivity in terms of which dynamics can be represented.

But almost all square matrices are diagonalizable over the complex plane: https://chiasme.wordpress.com/2013/09/03/almost-all-matrices-are-diagonalizable/

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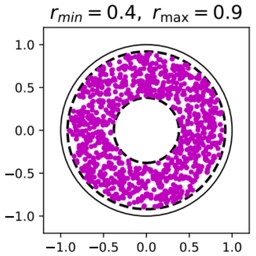

Figure 3 | *Eigenvalues of a diagonal matrix A with entries sampled using Lemma 3.2. For* $r_{\text{min}} = 0$, $r_{\text{max}} = 1$, *the distribution coincides with Glorot init. in the limit.*

Stability criteria:

- **To avoid exploding, discrete eigenvalues should be within the complex unit circle**

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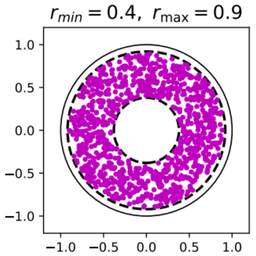

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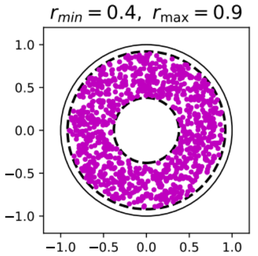

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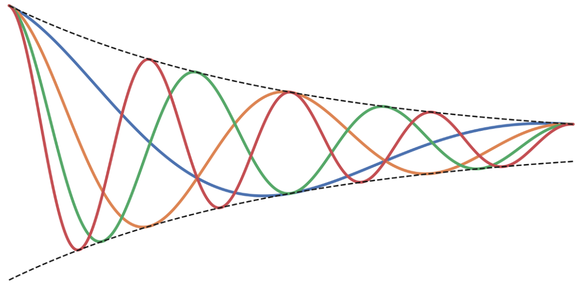
HiPPO initialization gives these nice properties with stable, slowly decaying eigenvalues



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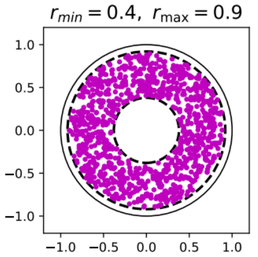

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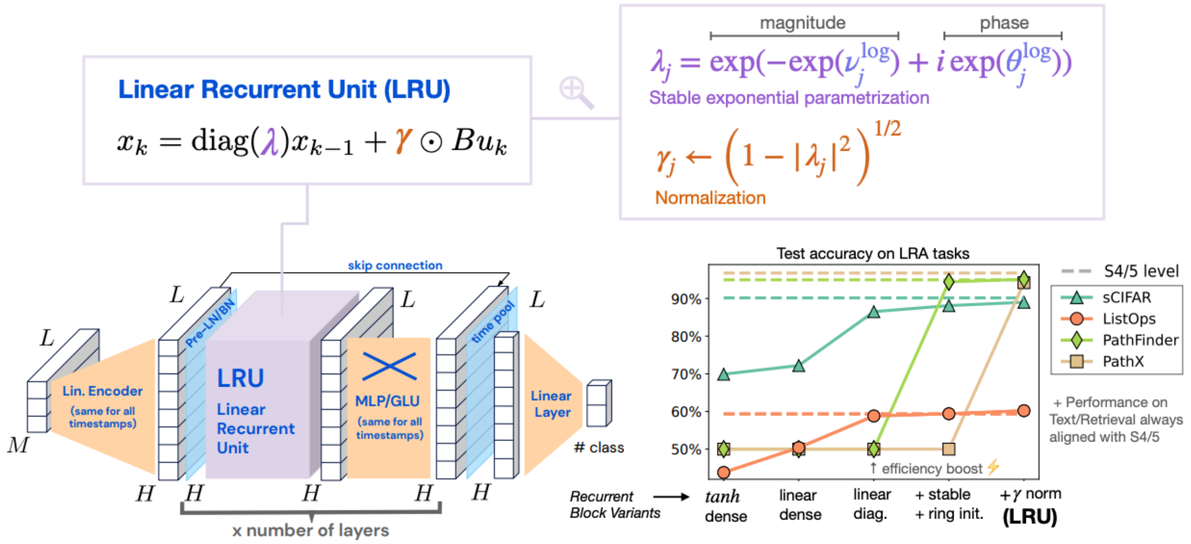

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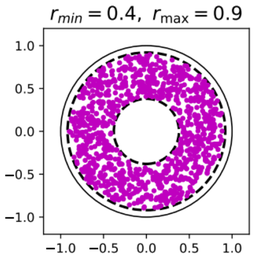

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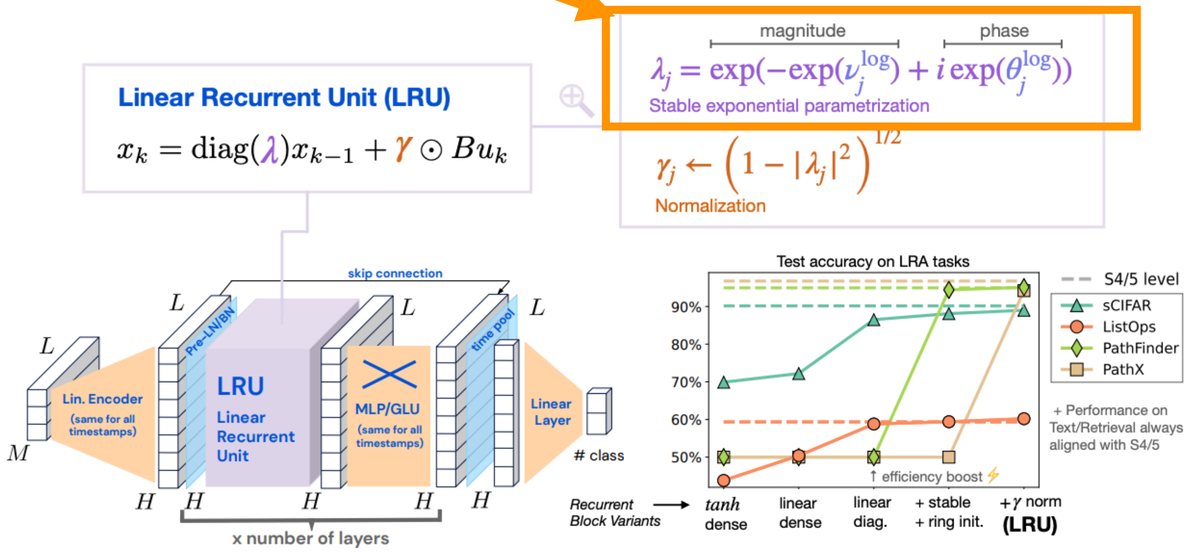
Takeaways:

- **Complex parameterization important for some problems**



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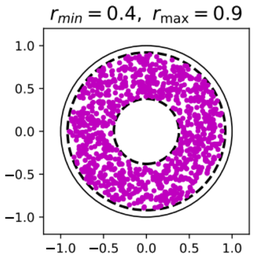

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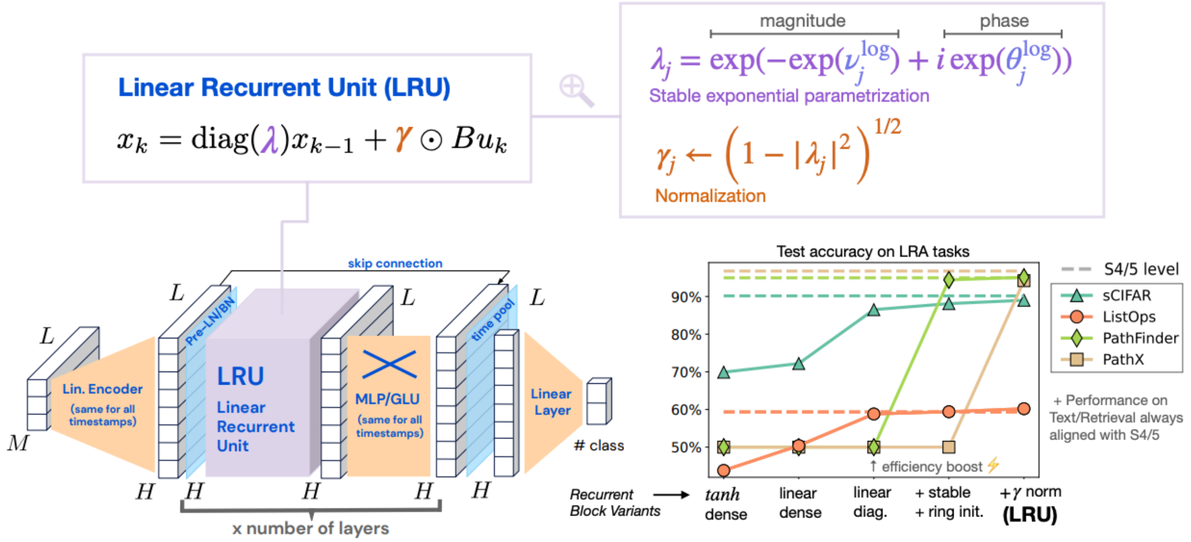
- **HiPPO+discretization gives a smart eigenvalue initialization**



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$$\mathbf{x}_k = e^{\mathbf{\Lambda}\Delta}\mathbf{x}_{k-1} + \mathbf{\Lambda}^{-1}(\overline{\mathbf{\Lambda}} - \mathbf{I})\mathbf{B}\mathbf{u}_k$$

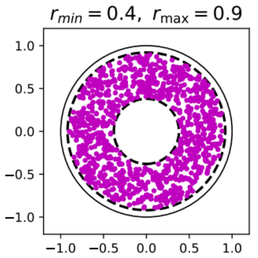
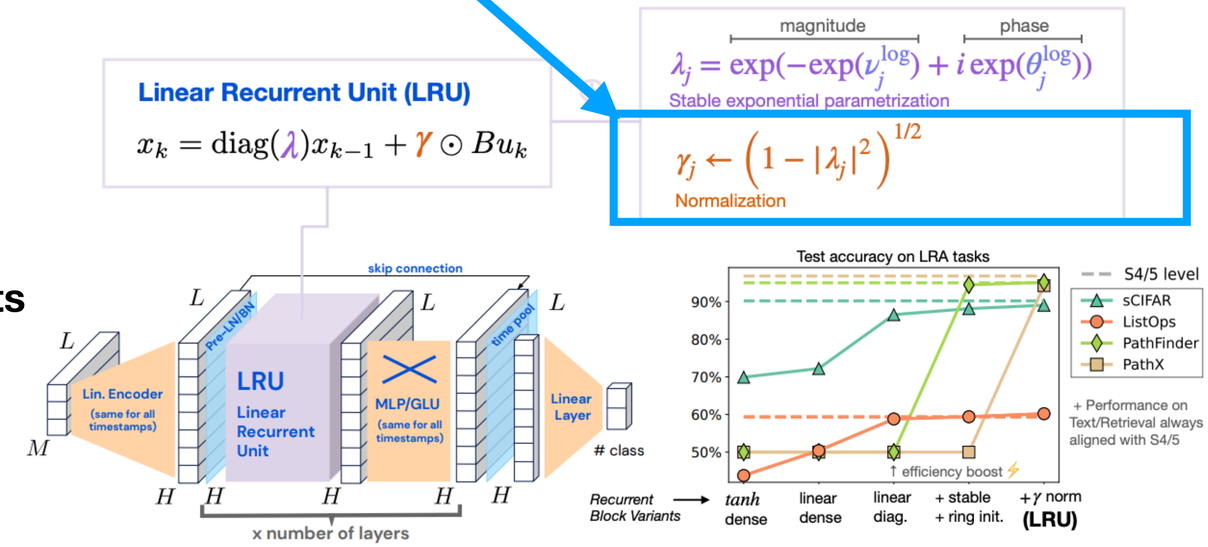


Figure 3 | *Eigenvalues of a diagonal matrix A with entries sampled using Lemma 3.2. For* $r_{\min} = 0$, $r_{\max} = 1$, *the distribution coincides with Glorot init. in the limit.*

LRU paper (Orvieto 2023), shows S4/S5 style linear RNNs can be parameterized without explicit discretization or HiPPO, but still achieve similar performance on benchmarks

Takeaways:

- **Complex parameterization important for some problems**
- **HiPPO+discretization gives a smart eigenvalue initialization**
- **Discretization provides normalizing effect on effective inputs**

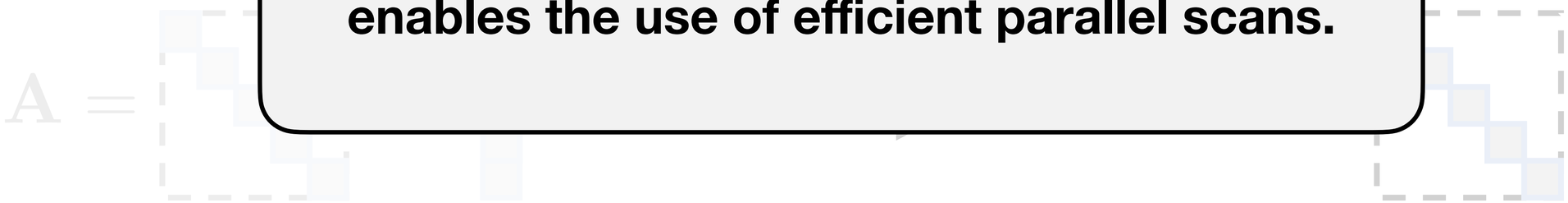


From S4 to S5: Diagonalized dynamics

Diagonal plus low-rank
state matrix

Diagonal
state matrix

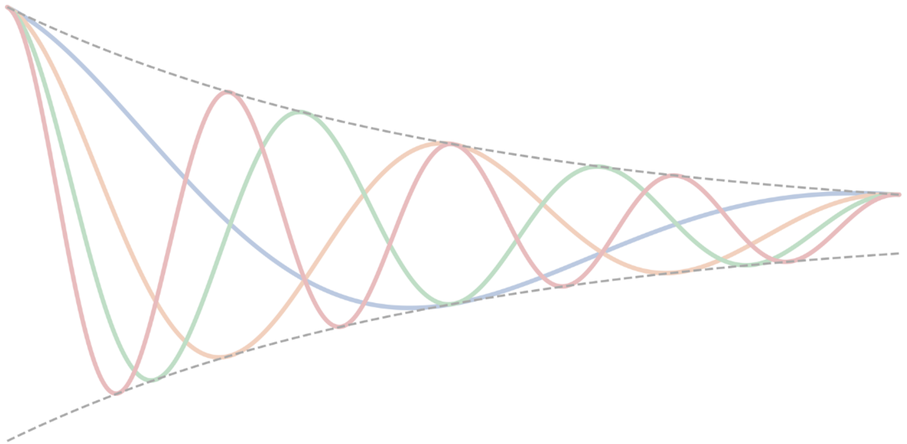
MIMO + diagonalized dynamics:
enables the use of efficient parallel scans.



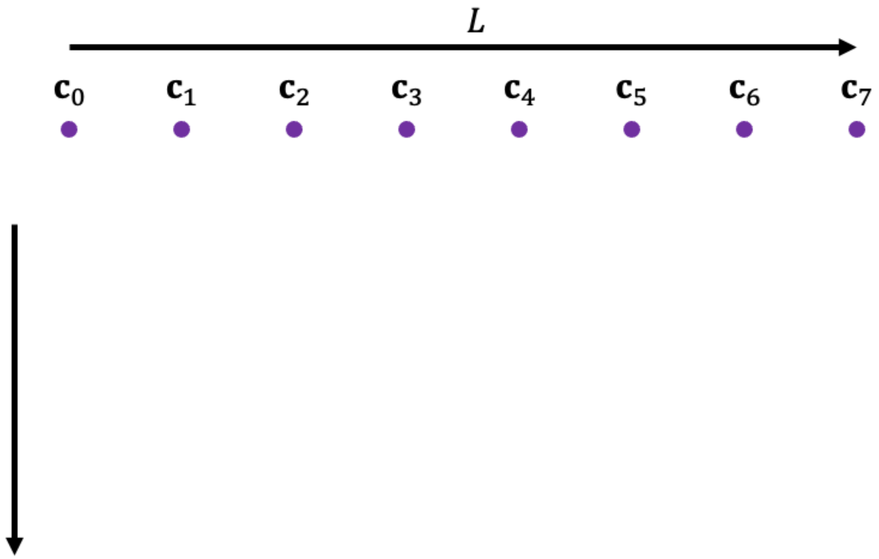
Similar findings to DSS (Gupta et al. 2022)
and S4D (Gu et al. 2022)

From S4 to S5: Fully Recurrent

Convolution



Parallel scan



Convolution limitations:

- **Requires time-invariant system**
- **Cannot access states**

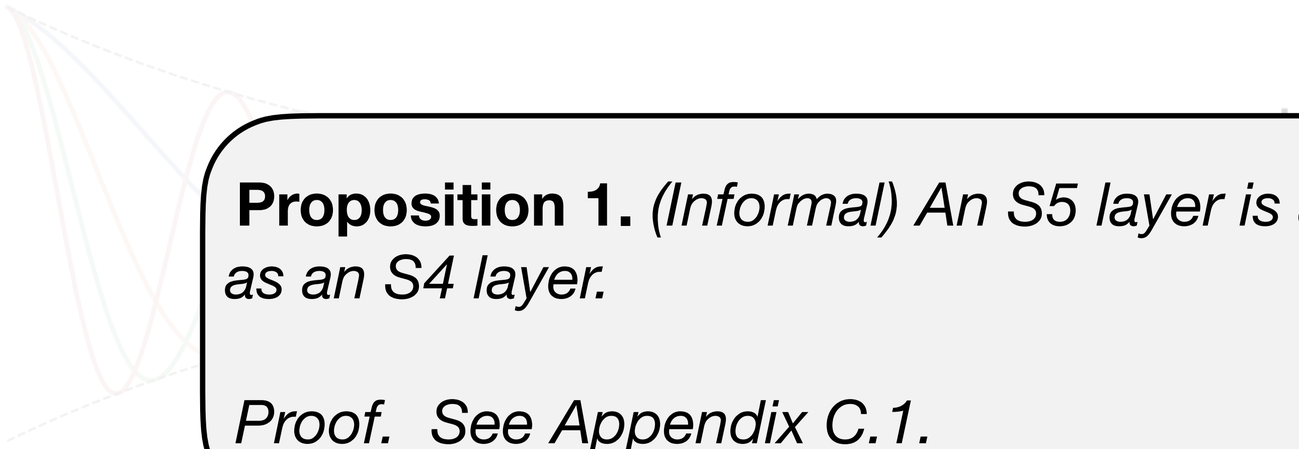
Scan allows:

- **Time-varying systems**
- **Access to states (parallel or autoregressive)**

From S4 to S5: Fully Recurrent

Convolution

Parallel scan



Proposition 1. *(Informal) An S5 layer is as efficient as an S4 layer.*

Proof. See Appendix C.1.

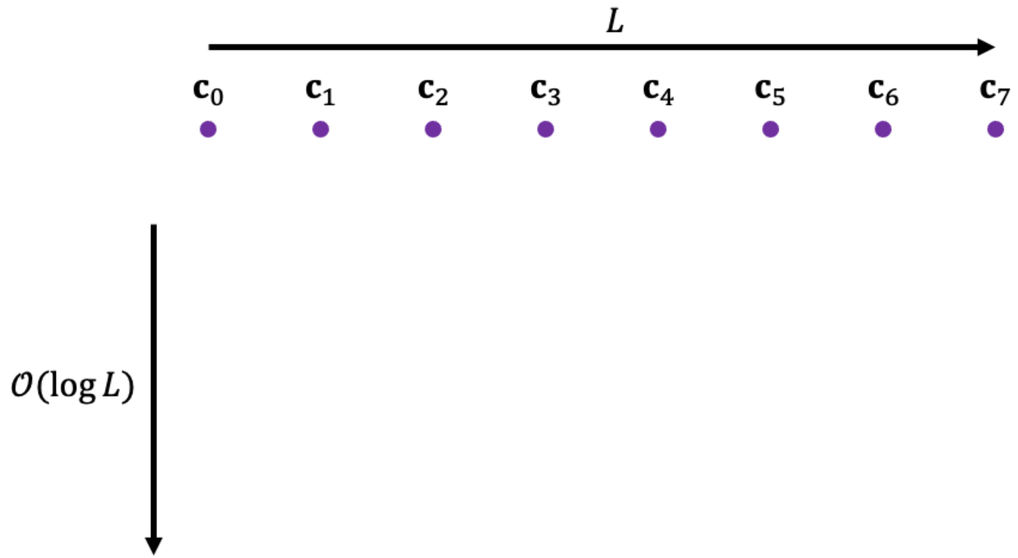
Convolution limitations:

- Requires time-invariant system
- Cannot access states

Scan allows:

- Time-varying systems
- Access to states (parallel or autoregressive)

From S4 to S5: Parallel Scans



From S4 to S5: Parallel Scans

Consider the scalar sequence: [a,b,c,d] and the addition operator +.
Performing a scan (all prefix sum) on this sequence using + returns the cumulative sum:

[a, a+b, a+b+c, a+b+c+d]

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[a, a+b, a+b+c, a+b+c+d]

Sequential Scan
(3 sequential steps required)

a

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Sequential Scan
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a+b+c
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Parallel Scan
(2 sequential steps required)

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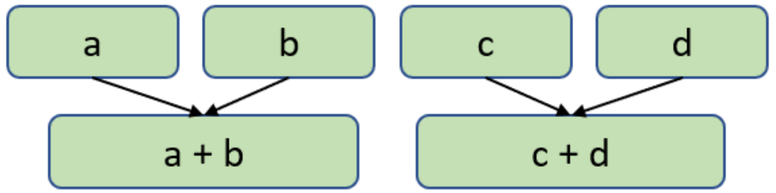

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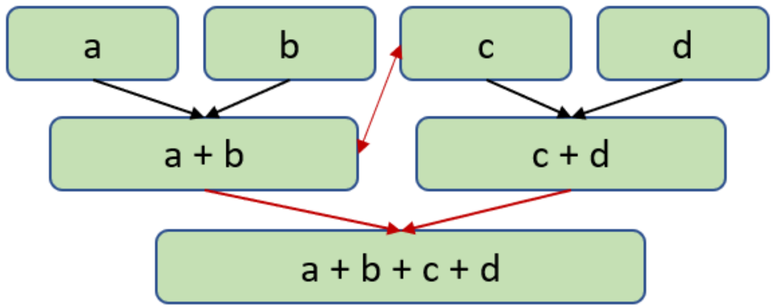

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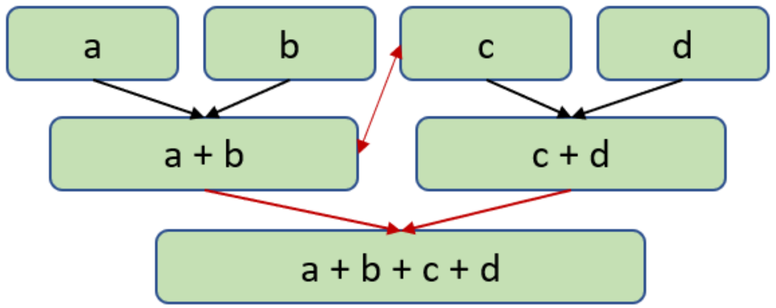

Given sufficient processors, number of sequential steps scales logarithmically in sequence length

From S4 to S5: Parallel Scans

$$x_k = \overline{\mathbf{A}} x_{k-1} + \overline{\mathbf{B}} u_k$$

Sequential Scan
(3 sequential steps required)

$$x_1 = \overline{\mathbf{B}} u_1$$

Parallel Scan
(2 sequential steps required)

From S4 to S5: Parallel Scans

$$x_k = \overline{\mathbf{A}} x_{k-1} + \overline{\mathbf{B}} u_k$$

Sequential Scan
(3 sequential steps required)

$$x_1 = \overline{\mathbf{B}} u_1$$
$$x_2 = \overline{\mathbf{AB}} u_1 + \overline{\mathbf{B}} u_2$$

Parallel Scan
(2 sequential steps required)

From S4 to S5: Parallel Scans

$$x_k = \overline{\mathbf{A}} x_{k-1} + \overline{\mathbf{B}} u_k$$

Sequential Scan
(3 sequential steps required)

$$x_1 = \overline{\mathbf{B}} u_1$$
$$x_2 = \overline{\mathbf{AB}} u_1 + \overline{\mathbf{B}} u_2$$
$$x_3 = \overline{\mathbf{A}}^2 \overline{\mathbf{B}} u_1 + \overline{\mathbf{AB}} u_2 + \overline{\mathbf{B}} u_3$$

Parallel Scan
(2 sequential steps required)

From S4 to S5: Parallel Scans

$$x_k = \overline{\mathbf{A}} x_{k-1} + \overline{\mathbf{B}} u_k$$

Sequential Scan
(3 sequential steps required)

$$x_1 = \overline{\mathbf{B}} u_1$$
$$x_2 = \overline{\mathbf{AB}} u_1 + \overline{\mathbf{B}} u_2$$
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$$x_4 = \overline{\mathbf{A}}^3 \overline{\mathbf{B}} u_1 + \overline{\mathbf{A}}^2 \overline{\mathbf{B}} u_2 + \overline{\mathbf{AB}} u_3 + \overline{\mathbf{B}} u_4$$

Parallel Scan
(2 sequential steps required)

From S4 to S5: Parallel Scans

$$x_k = \overline{\mathbf{A}} x_{k-1} + \overline{\mathbf{B}} u_k$$

Binary associative operator: $q_i \bullet q_j := (q_{j,a} \odot q_{i,a},\ \ q_{j,a} \otimes q_{i,b} + q_{j,b})$

Sequential Scan
(3 sequential steps required)

$$x_1 = \overline{\mathbf{B}} u_1$$
$$x_2 = \overline{\mathbf{AB}} u_1 + \overline{\mathbf{B}} u_2$$
$$x_3 = \overline{\mathbf{A}}^2 \overline{\mathbf{B}} u_1 + \overline{\mathbf{AB}} u_2 + \overline{\mathbf{B}} u_3$$
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Parallel Scan
(2 sequential steps required)

(A, Bu_1) (A, Bu_2) (A, Bu_3) (A, Bu_4)

From S4 to S5: Parallel Scans

$$x_k = \overline{\mathbf{A}} x_{k-1} + \overline{\mathbf{B}} u_k$$

Binary associative operator: $q_i \bullet q_j := (q_{j,a} \odot q_{i,a},\ \ q_{j,a} \otimes q_{i,b} + q_{j,b})$

Sequential Scan
(3 sequential steps required)

$$x_1 = \overline{\mathbf{B}} u_1$$
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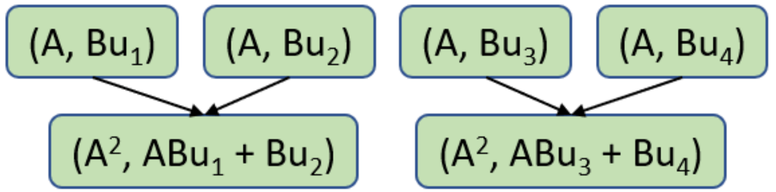
Parallel Scan
(2 sequential steps required)



From S4 to S5: Parallel Scans

$$x_k = \overline{\mathbf{A}} x_{k-1} + \overline{\mathbf{B}} u_k$$

Binary associative operator: $q_i \bullet q_j := (q_{j,a} \odot q_{i,a},\ \ q_{j,a} \otimes q_{i,b} + q_{j,b})$

Sequential Scan
(3 sequential steps required)

$$x_1 = \overline{\mathbf{B}} u_1$$
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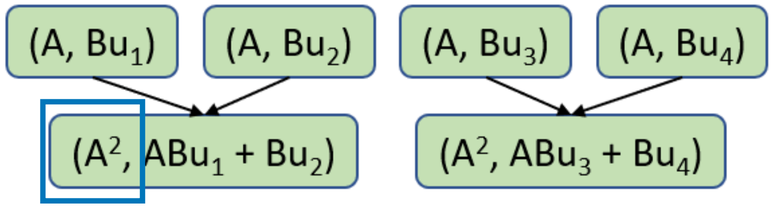
$$x_3 = \overline{\mathbf{A}}^2 \overline{\mathbf{B}} u_1 + \overline{\mathbf{AB}} u_2 + \overline{\mathbf{B}} u_3$$
$$x_4 = \overline{\mathbf{A}}^3 \overline{\mathbf{B}} u_1 + \overline{\mathbf{A}}^2 \overline{\mathbf{B}} u_2 + \overline{\mathbf{AB}} u_3 + \overline{\mathbf{B}} u_4$$

Parallel Scan
(2 sequential steps required)



Note: matrix-matrix multiplication,
this is why diagonalization is important to avoid cubic cost!

From S4 to S5: Parallel Scans

$$x_k = \overline{\mathbf{A}} x_{k-1} + \overline{\mathbf{B}} u_k$$

Binary associative operator: $q_i \bullet q_j := (q_{j,a} \odot q_{i,a},\ \ q_{j,a} \otimes q_{i,b} + q_{j,b})$

Sequential Scan
(3 sequential steps required)

$$x_1 = \overline{\mathbf{B}} u_1$$
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Parallel Scan
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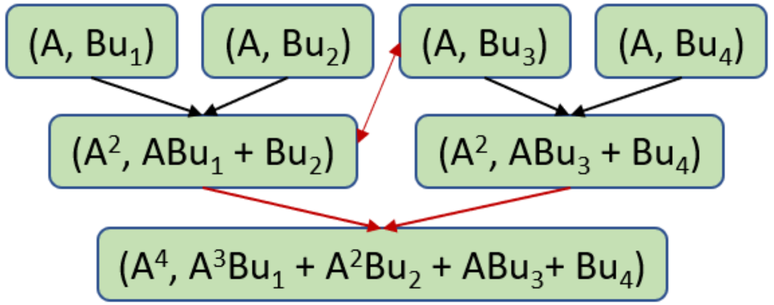


Given sufficient processors, number of sequential steps scales logarithmically in sequence length

S5 retains S4's high performance

Long Range Arena

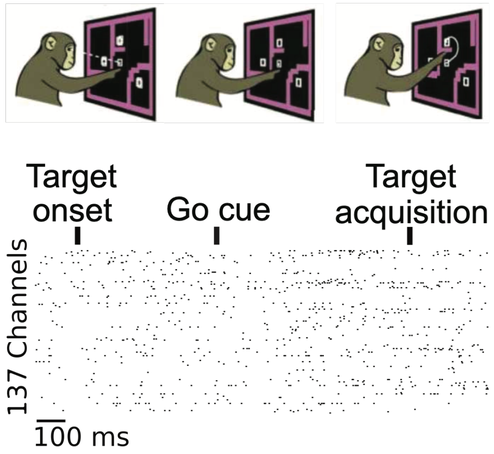
Model (Input length)	ListOps (2,048)	Text (4,096)	Retrieval (4,000)	Image (1,024)	Pathfinder (1,024)	Path-X (16,384)	Avg.
Transformer	36.37	64.27	57.46	42.44	71.40	✗	53.66
S4D-LegS	<u>60.47</u>	86.18	89.46	<u>88.19</u>	93.06	91.95	84.89
S4-LegS	59.60	<u>86.82</u>	<u>90.90</u>	**88.65**	<u>94.20</u>	<u>96.35</u>	<u>86.09</u>
S5	**62.15**	**89.31**	**91.40**	88.00	**95.33**	**98.58**	**87.46**

S5 retains S4's high performance

Speech Commands 35-way Raw Speech Classification

Model (Input length)	Parameters	16kHz (16,000)	8kHz (8,000)
InceptionNet	481K	61.24	05.18
ResNet-1	216K	77.86	08.74
XResNet-50	904K	83.01	07.72
ConvNet	26.2M	95.51	07.26
S4-LegS	307K	<u>96.08</u>	<u>91.32</u>
S4D-LegS	306K	95.83	91.08
S5	280K	**96.52**	**94.53**

Neural Latents Benchmark



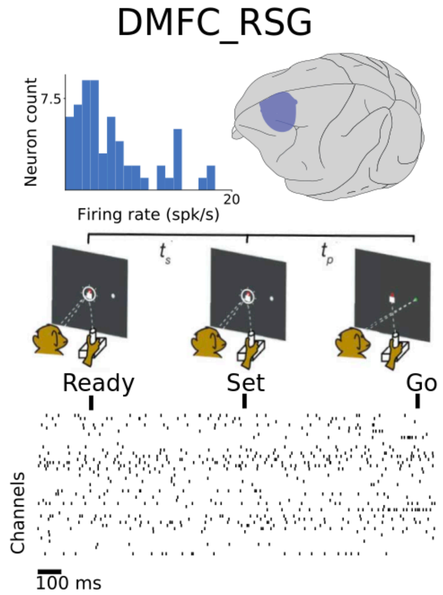
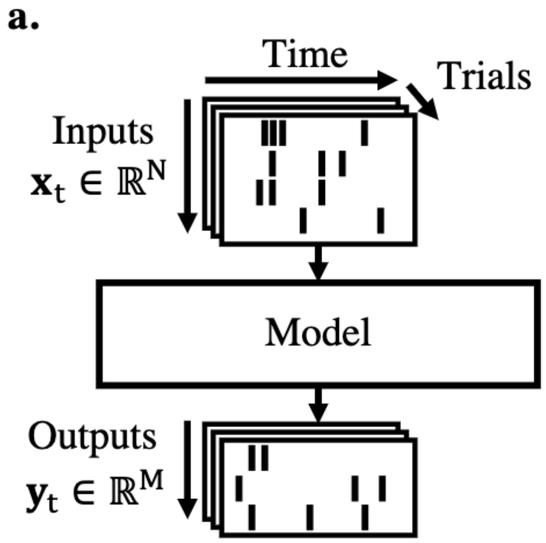


Table 1: Co-smoothing (in units of bits-per-spike) metric on `MC_Maze` and `DMFC_RSG` benchmarks [Pei *et al.*, 2021] for S5 compared to SOTA methods. Note: we exclude ensemble methods and only consider single models.

Method	`MC_Maze` (↑)	`DMFC_RSG` (↑)
S5 (Ours)	**0.3826**	**0.1981**
SSLFADS	0.3748	N/A
STNDT	0.3691	0.1859
iLQR-VAE	0.3559	N/A
Neural RoBERTa	0.3551	N/A
RNNf	0.3382	0.1781
AutoLFADS	0.3364	0.1829
MINT	0.3304	0.1821
NDT	0.3229	0.1720
SLDS	0.2249	0.1243

w/ Hyun Lee



https://eval.ai/web/challenges/challenge-page/1256/leaderboard/3188

ConvSSMs: ConvRNN spatiotemporal inductive biases + SSM efficiency and long range modeling

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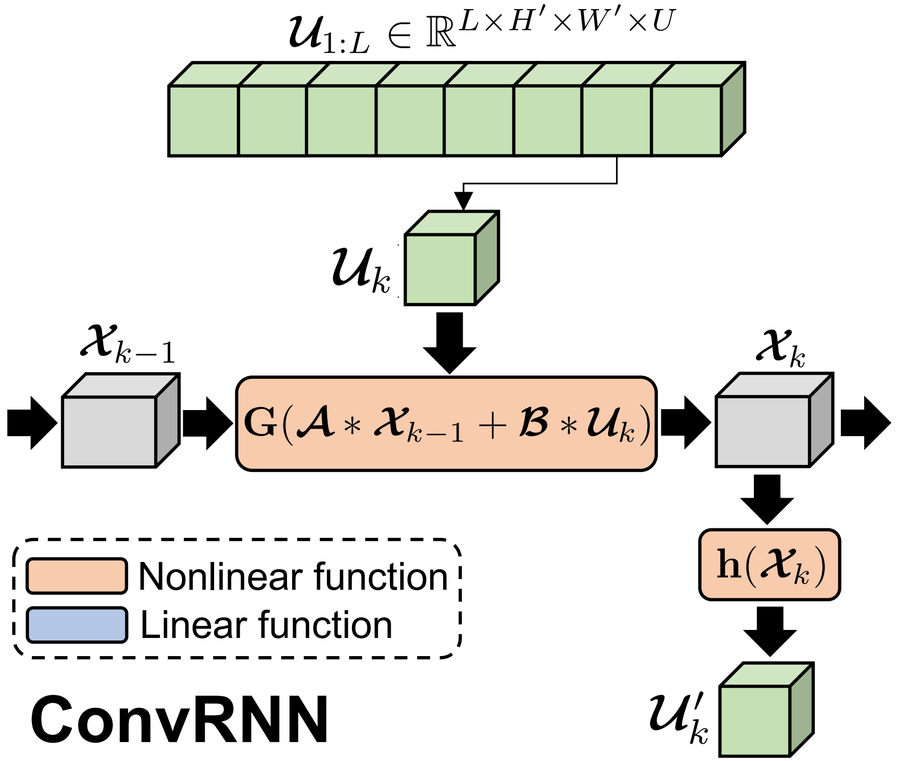
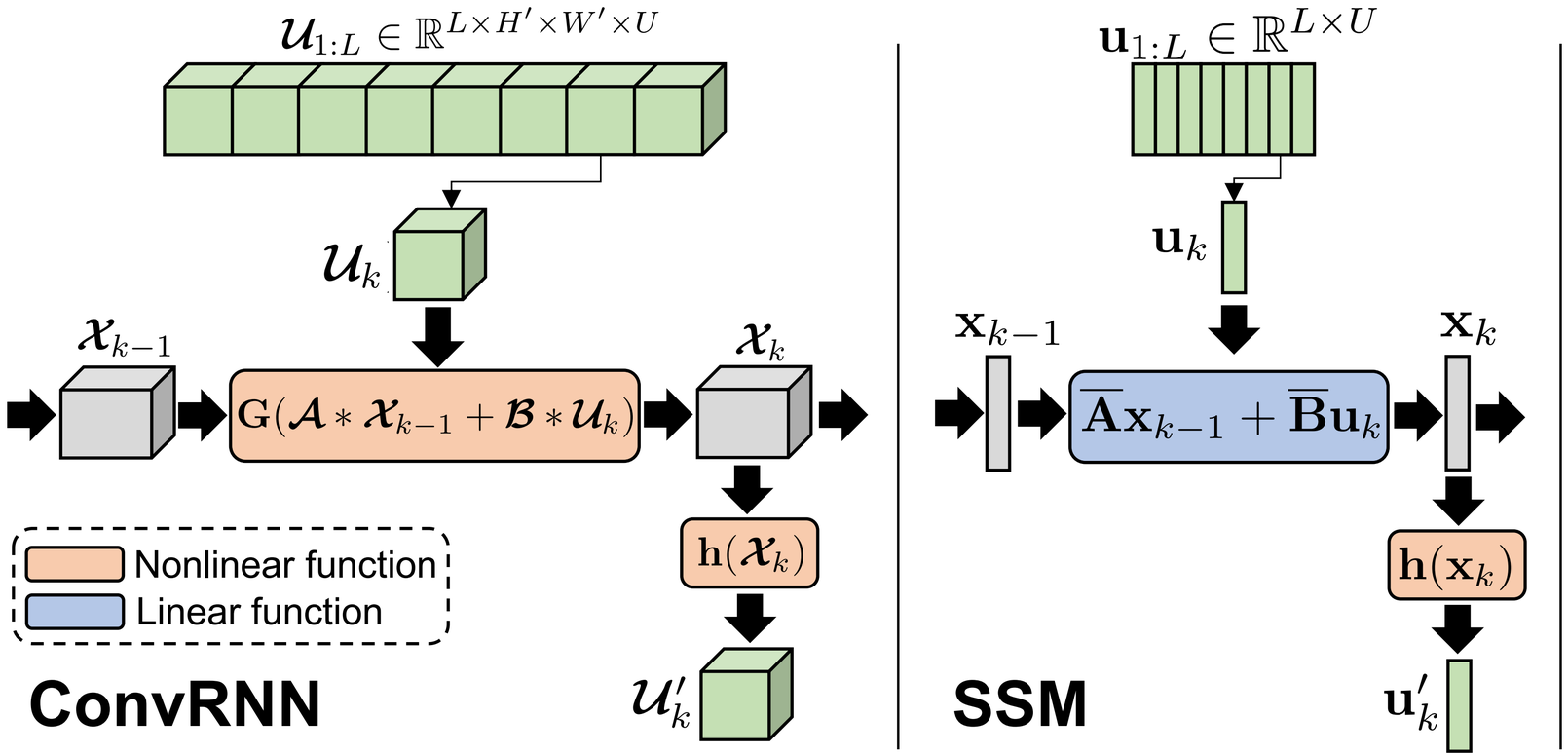


ConvRNN

✓ **Spatiotemporal modeling**
✓ **Fast autoregressive generation**
X Slow, sequential training
X Vanishing gradients

Smith et al. Convolutional State Space Models for Long-Range Spatiotemporal Modeling. 2023.

ConvSSMs: ConvRNN spatiotemporal inductive biases + SSM efficiency and long range modeling



- ✓ **Spatiotemporal modeling**
- ✓ **Fast autoregressive generation**
- X Slow, sequential training
- X Vanishing gradients

- ✓ **Efficient, parallel training**
- ✓ **Fast autoregressive generation**
- ✓ **Captures long-range dependencies**
- X Designed for vector-valued sequences

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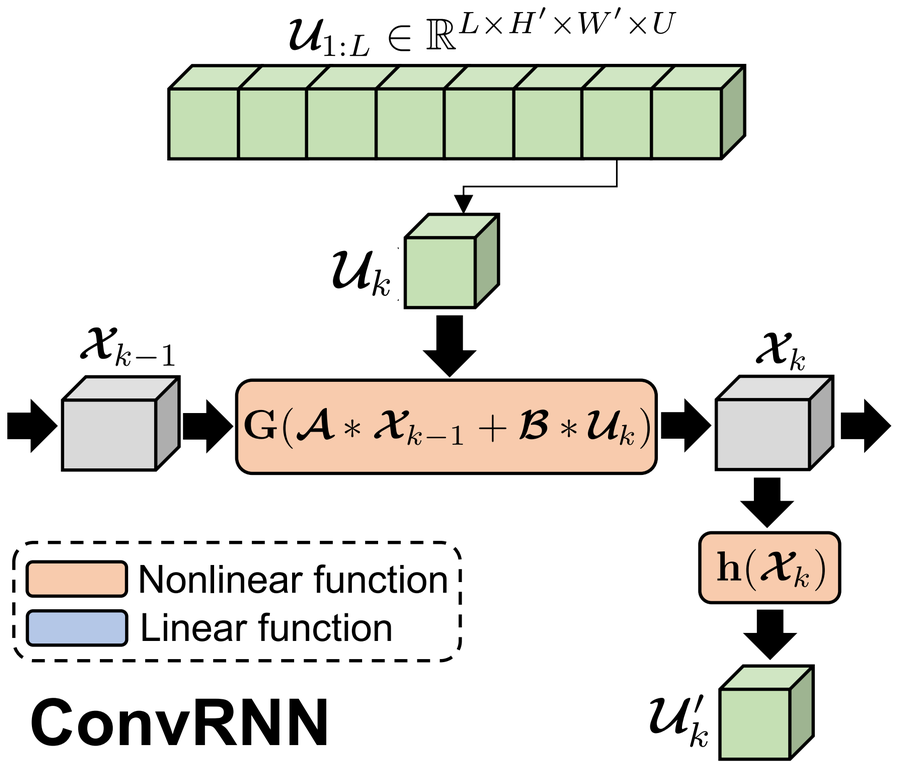
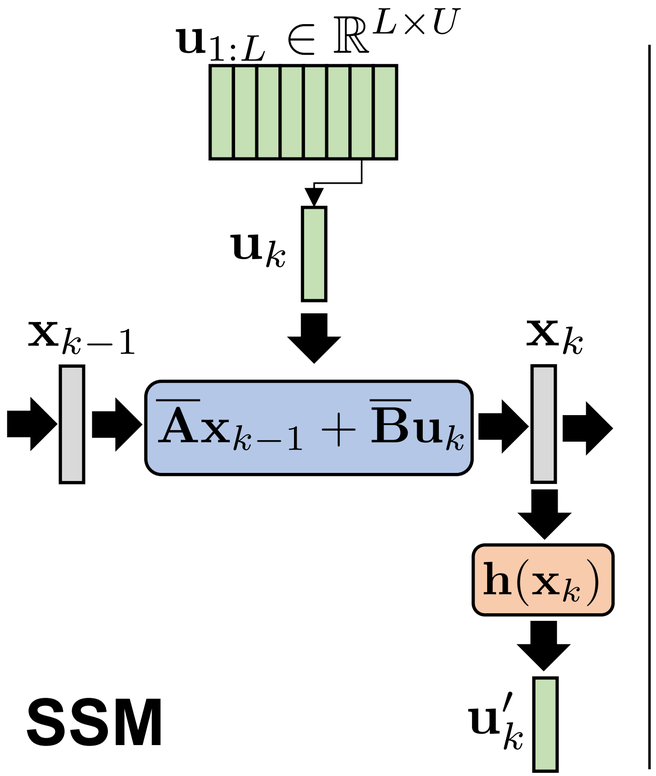
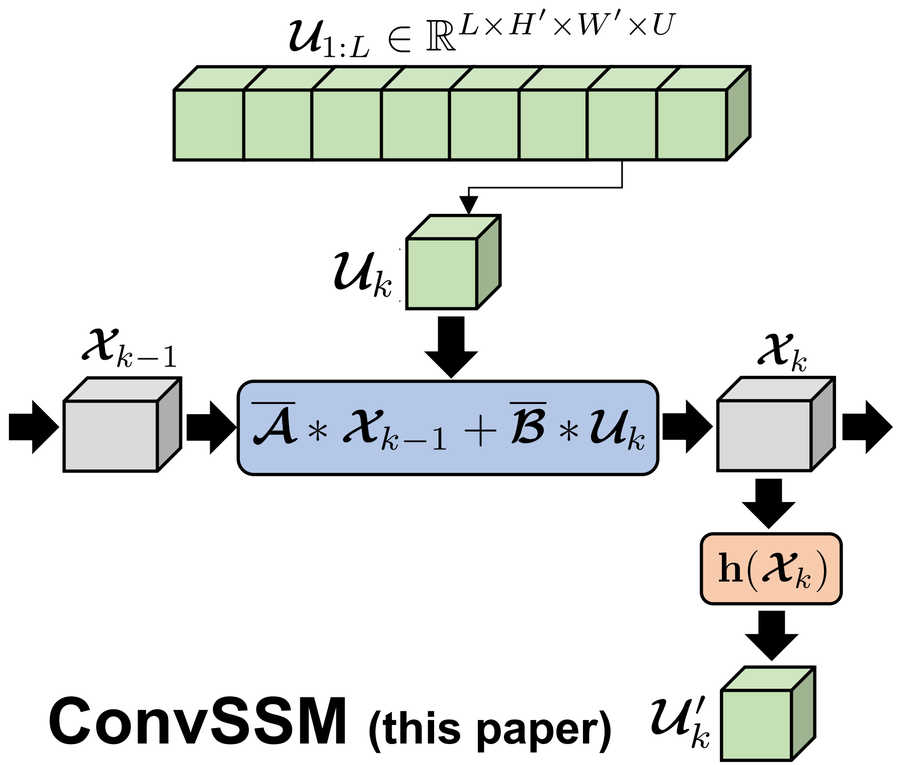


ConvRNN



SSM



ConvSSM (this paper)

- ✓ Spatiotemporal modeling
- ✓ Fast autoregressive generation
- X Slow, sequential training
- X Vanishing gradients

- ✓ Efficient, parallel training
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- ✓ Efficient, parallel training
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- ✓ Spatiotemporal modeling

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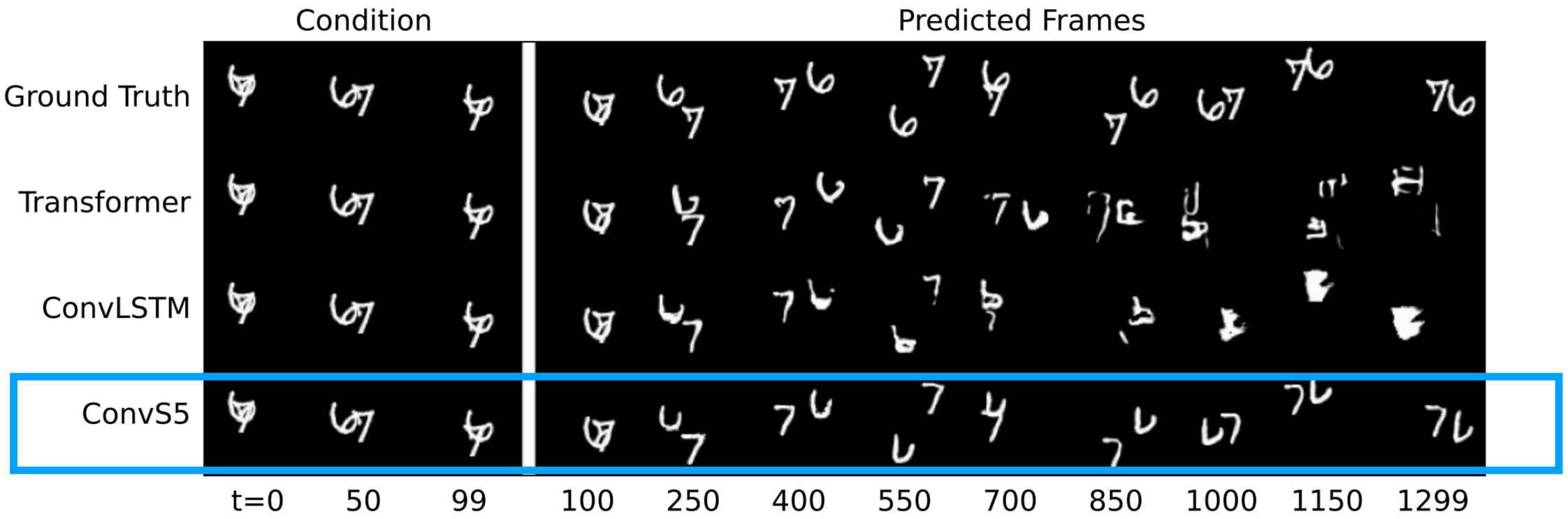

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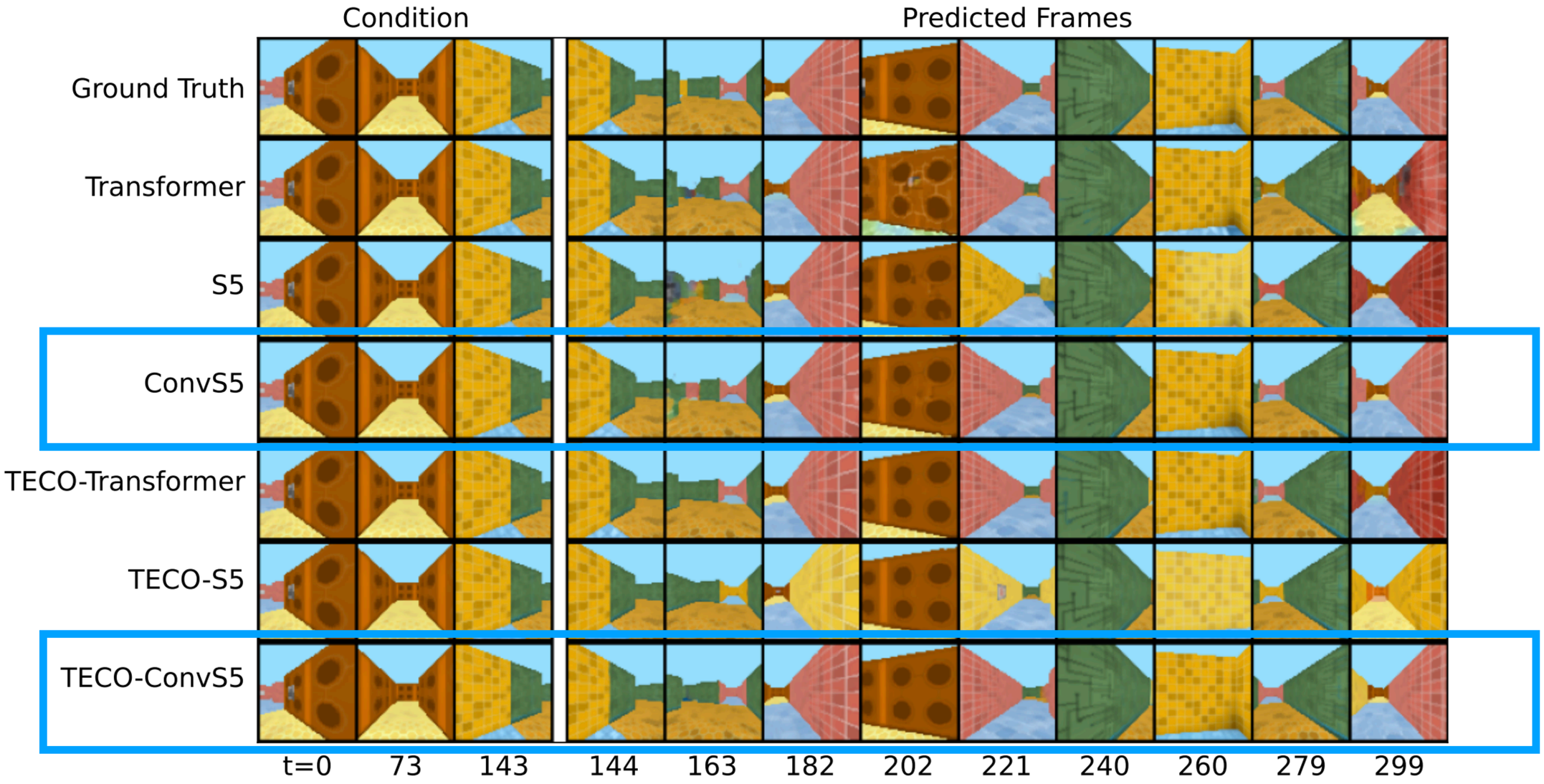

Smith et al. Convolutional State Space Models for Long-Range Spatiotemporal Modeling. 2023.

S5 enables new capabilities

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S5 can use linear time-varying (LTV) state space models:

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

S5 enables new capabilities

S5 can use linear time-varying (LTV) state space models:

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

- **Context dependent dynamics**

$$\mathbf{x}_k = \overline{\mathbf{A}}(\mathbf{u}_{1:k})\mathbf{x}_{k-1} + \overline{\mathbf{B}}(\mathbf{u}_{1:k})\mathbf{u}_k$$

S5 enables new capabilities

S5 can use linear time-varying (LTV) state space models:

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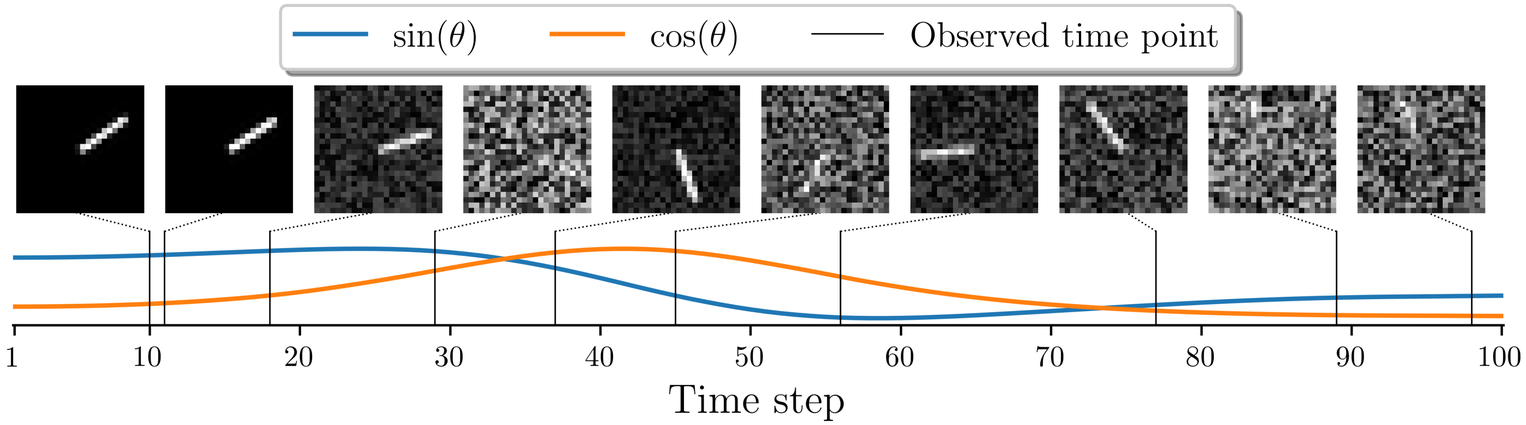
- **Context dependent dynamics**

$$\mathbf{x}_k = \overline{\mathbf{A}}(\mathbf{u}_{1:k})\mathbf{x}_{k-1} + \overline{\mathbf{B}}(\mathbf{u}_{1:k})\mathbf{u}_k$$

- **Irregularly sampled time-series**

$$\mathbf{x}_k = \overline{\mathbf{A}}(\boldsymbol{\Delta}_k)\mathbf{x}_{k-1} + \overline{\mathbf{B}}(\boldsymbol{\Delta}_k)\mathbf{u}_k$$

LTV example: Irregularly sampled time series



Model	Relative speed ↑	Regression MSE ($\times 10^{-3}$) ↓
mTAND	<u>12x</u>	65.64 (4.05)
RKN	1.9x	8.43 (0.61)
RKN-Δ_t	1.9x	5.09 (0.40)
GRU	3.0x	9.44 (1.00)
GRU-Δ_t	3.0x	5.44 (0.99)
Latent ODE	0.7x	15.70 (2.85)
ODE-RNN	1.0x	7.26 (0.41)
GRU-ODE-B	0.6x	9.78 (3.40)
f-CRU	1.2x	6.16 (0.88)
CRU	1.0x	4.63 (1.07)
CRU (our run)	1.0x	<u>3.94</u> (0.21)
S5	**86x**	**3.41** (0.27)

Agenda

- Introduction, motivation, prior approaches
- Linear state space models (SSMs) overview
- S4, convolutions, parameterization
- S5, diagonalization, parallel scans
- **S6/Mamba/Hawk/Griffin, data-dependent dynamics**
- Challenges and opportunities

SSMs/RNNs vs Softmax Attention on Language

Deep SSMs, such as S4 and S5, mostly using LTI systems, have proven effective in a variety of data modalities: speech, image, video, reinforcement learning etc.

Arora et al. Zoology: Measuring and Improving Recall in Efficient Language Models. 2023.

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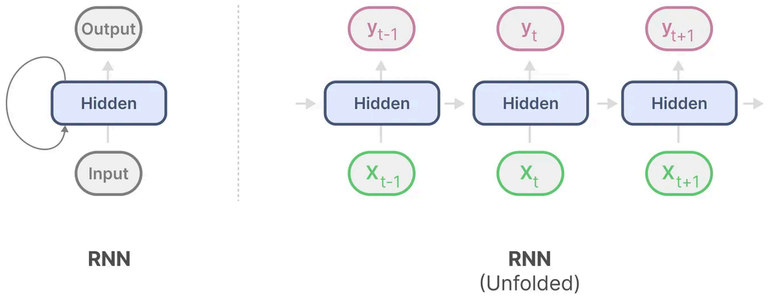
But exact, lossless recall is difficult for fixed state models such as SSMs/RNNs compared to Softmax Attention.



RNN

RNN
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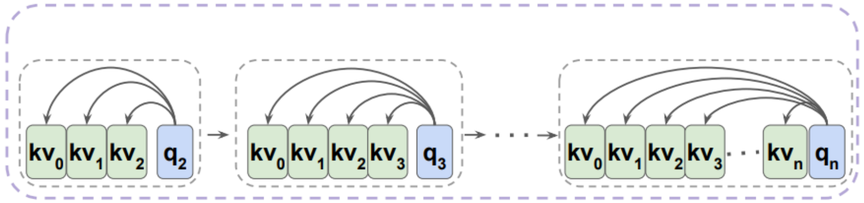
Vs.



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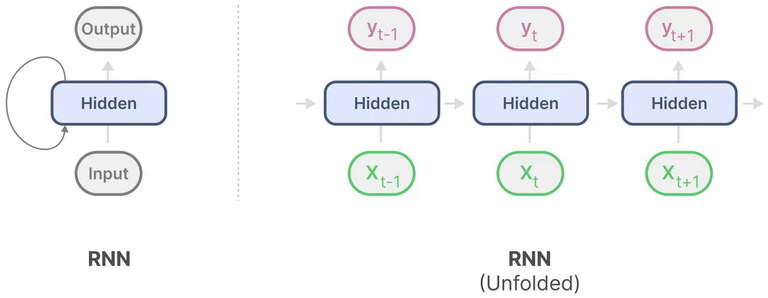

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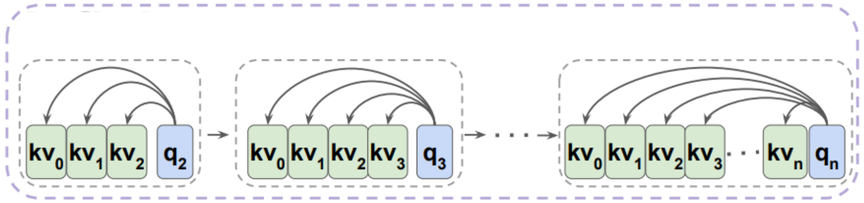
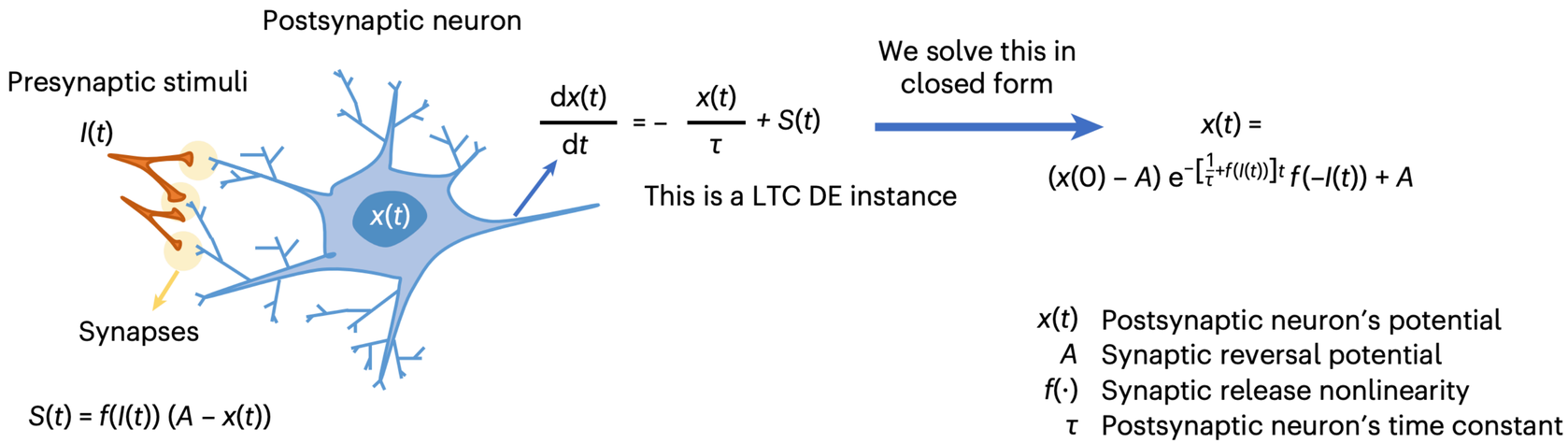


Can we make better use of this fixed state with linear time-varying systems (LTV)?

Arora et al. Zoology: Measuring and Improving Recall in Efficient Language Models. 2023.

Liquid Time-Constant Networks

$$\frac{d\mathbf{x}(t)}{dt} = -\left[\frac{1}{\tau} + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)\right]\mathbf{x}(t) + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)A.$$



Hasani and Lechner et al. Liquid time constant networks 2021.
Hasani and Lechner et al. Closed-form continuous-time neural networks. 2022.

LTV example: Liquid S4

$$x_k = \left(\overline{\mathbf{A}} + \overline{\mathbf{B}}\, u_k\right) x_{k-1} + \overline{\mathbf{B}}\, u_k, \qquad y_k = \overline{\mathbf{C}}\, x_k$$

$$\begin{aligned}
x_0 &= \overline{\mathbf{B}} u_0, \quad y_0 = \overline{\mathbf{CB}} u_0 \\
x_1 &= \overline{\mathbf{AB}} u_0 + \overline{\mathbf{B}} u_1 + \overline{\mathbf{B}}^2 u_0 u_1, \quad y_1 = \overline{\mathbf{CAB}} u_0 + \overline{\mathbf{CB}} u_1 + \overline{\mathbf{CB}}^2 u_0 u_1 \\
x_2 &= \overline{\mathbf{A}}^2 \overline{\mathbf{B}} u_0 + \overline{\mathbf{AB}} u_1 + \overline{\mathbf{B}} u_2 + \overline{\mathbf{AB}}^2 u_0 u_1 + \overline{\mathbf{AB}}^2 u_0 u_2 + \overline{\mathbf{B}}^2 u_1 u_2 + \overline{\mathbf{B}}^3 u_0 u_1 u_2 \\
y_2 &= \overline{\mathbf{CA}}^2 \overline{\mathbf{B}} u_0 + \overline{\mathbf{CAB}} u_1 + \overline{\mathbf{CB}} u_2 + \overline{\mathbf{CAB}}^2 u_0 u_1 + \overline{\mathbf{CAB}}^2 u_0 u_2 + \overline{\mathbf{CB}}^2 u_1 u_2 + \overline{\mathbf{CB}}^3 u_0 u_1 u_2, \quad \ldots
\end{aligned}$$

- **Generally, LTV systems cannot be computed using convolutions.**
- **But Liquid-S4 work shows how this specific LTV form can be computed efficiently using convolutions.**

Hasani and Lechner et al. Liquid structural state-space models. 2022.

Linear time-varying systems: S6/Mamba

$$\mathbf{x}_k = \overline{\mathbf{A}}(\mathbf{u}_{1:k})\mathbf{x}_{k-1} + \overline{\mathbf{B}}(\mathbf{u}_{1:k})\mathbf{u}_k$$

Gu and Dao. Mamba: Linear-Time Sequence Modeling with Selective State Spaces 2023

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- **Time-varying, data-dependent SSM parameters, similar to Liquid-S4, but more general.**

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Output: $y : (\mathtt{B}, \mathtt{L}, \mathtt{D})$
1: $\boldsymbol{A} : (\mathtt{D}, \mathtt{N}) \leftarrow$ Parameter
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▷ Time-invariant: recurrence or convolution
7: **return** y

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Gu and Dao. Mamba: Linear-Time Sequence Modeling with Selective State Spaces 2023

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Time varying dynamics allows for ignoring irrelevant inputs, or forgetting information that is no longer important to remember.

Gu and Dao. Mamba: Linear-Time Sequence Modeling with Selective State Spaces 2023

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Gu and Dao. Mamba: Linear-Time Sequence Modeling with Selective State Spaces 2023

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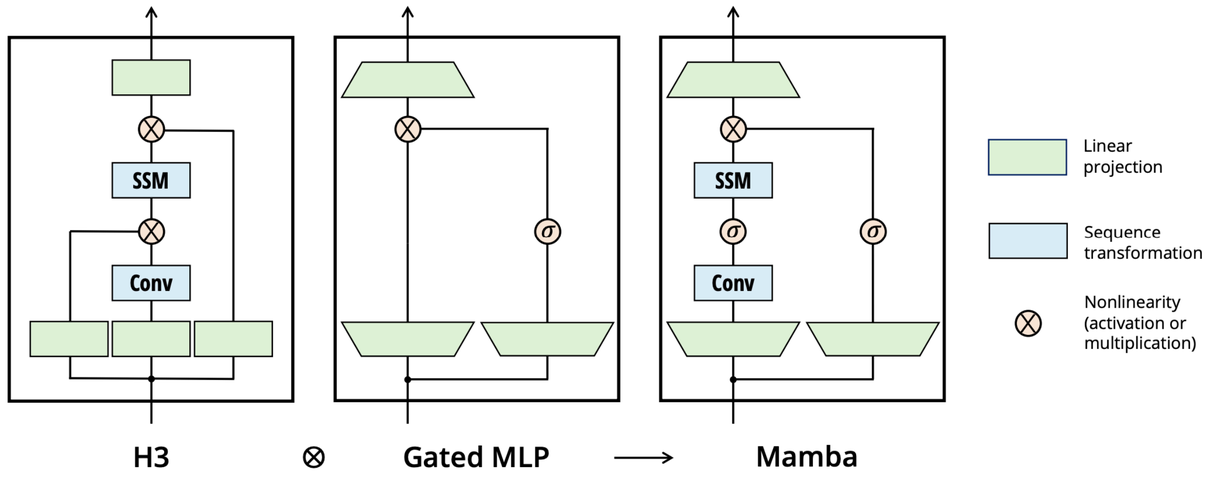
Block design:



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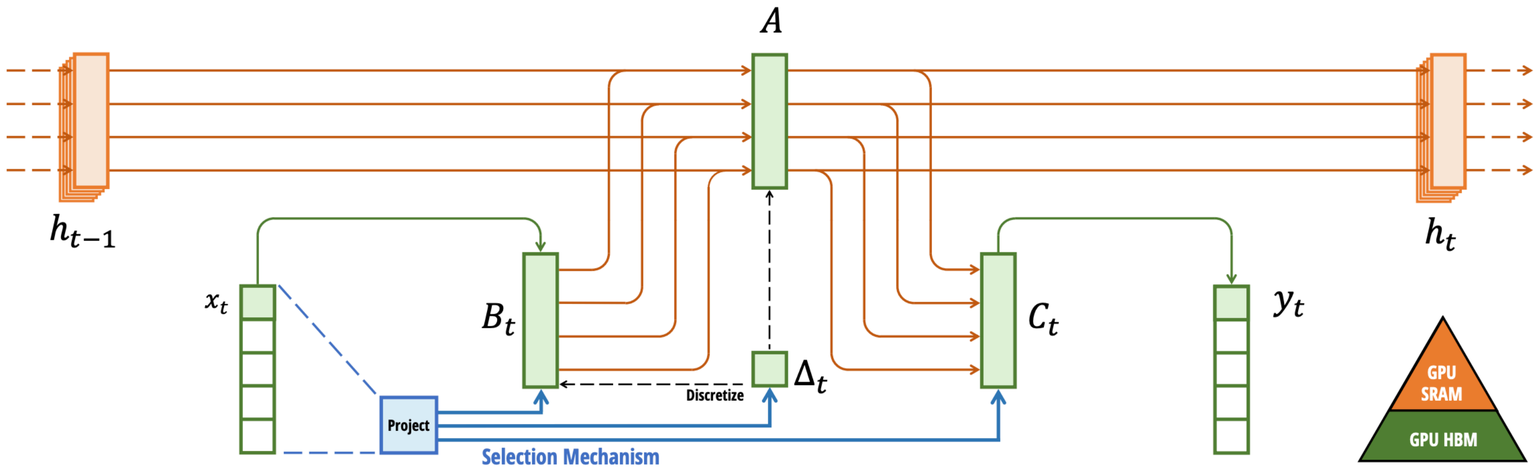
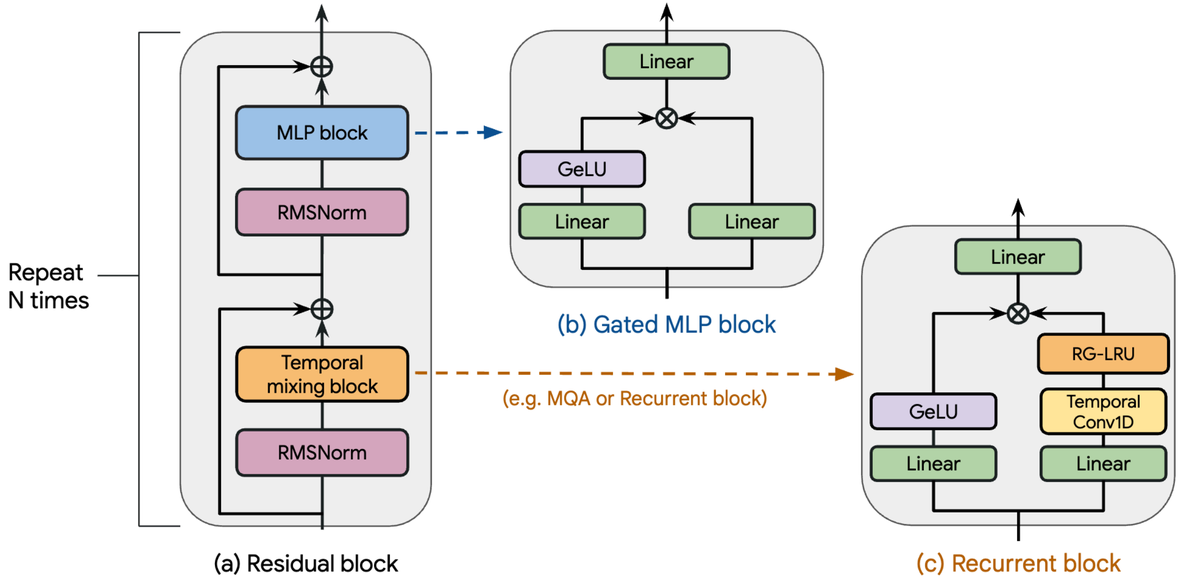
- Scans are limited by memory bandwidth
- This scan loads SSM params from slow HBM to fast SRAM, performs the discretization and recurrence in SRAM, and then writes outputs back to HBM



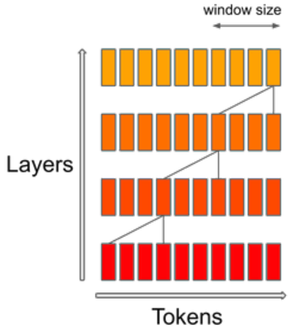
Gu and Dao. Mamba: Linear-Time Sequence Modeling with Selective State Spaces 2023

Linear time-varying systems: Hawk/Griffin

$$
\begin{aligned}
r_t &= \sigma(W_a x_t + b_a), \quad \textit{recurrence gate} \\
i_t &= \sigma(W_x x_t + b_x), \quad \textit{input gate} \\
a_t &= a^{cr_t}, \\
h_t &= a_t \odot h_{t-1} + \sqrt{1 - a_t^2} \odot (i_t \odot x_t).
\end{aligned}
$$



Griffin/Recurrent Gemma: Also include variant that includes sliding window attention in some layers



De et al. Griffin: Mixing Gated Linear Recurrences with Local Attention for Efficient Language Models 2024

Linear time-varying systems: Language Results

Model Type	Model Size	Training Tokens	MMLU	HellaSwag	PIQA	WinoGrande	ARC-E	ARC-C	Average
Mamba	3B	600B	26.2	71.0	78.1	65.9	68.2	41.7	58.5
Llama-2	7B	2T	45.3	77.2	78.8	69.2	75.2	45.9	65.3
	13B	2T	**54.8**	80.7	80.5	72.8	77.3	49.4	69.3
MQA Transformer (Baseline)	1B	300B	28.9	64.8	75.0	62.0	60.2	35.4	54.4
	3B	300B	31.7	71.0	77.6	66.1	68.1	39.2	59.0
	6B	300B	38.9	77.0	79.5	70.4	74.1	45.2	64.2
Hawk	1B	300B	29.7	63.3	76.1	57.2	60.6	34.6	53.6
	3B	300B	31.3	71.7	78.8	66.5	68.4	40.2	59.5
	7B	300B	35.0	77.6	80.0	69.9	74.4	45.9	63.8
Griffin	1B	300B	29.5	67.2	77.4	65.2	67.0	36.9	57.2
	3B	300B	32.6	73.5	78.1	67.2	71.5	41.4	60.7
	7B	300B	39.3	78.6	81.0	72.6	75.4	47.9	65.8
	14B	300B	49.5	**81.4**	**81.8**	**74.1**	**79.1**	**50.8**	**69.5**

These models can sometimes match Transformer perplexity and standard academic benchmark results

De et al. Griffin: Mixing Gated Linear Recurrences with Local Attention for Efficient Language Models 2024

Linear time-varying systems: Language Results

Model Type	Model Size	Training Tokens	MMLU	HellaSwag	PIQA	WinoGrande	ARC-E	ARC-C	Average
Mamba	3B	600B	26.2	71.0	78.1	65.9	68.2	41.7	58.5
Llama-2	7B	2T	45.3	77.2	78.8	69.2	75.2	45.9	65.3
	13B	2T	**54.8**	80.7	80.5	72.8	77.3	49.4	69.3
MQA Transformer (Baseline)	1B	300B	28.9	64.8	75.0	62.0	60.2	35.4	54.4
	3B	300B	31.7	71.0	77.6	66.1	68.1	39.2	59.0
	6B	300B	38.9	77.0	79.5	70.4	74.1	45.2	64.2
Hawk	1B	300B	29.7	63.3	76.1	57.2	60.6	34.6	53.6
	3B	300B	31.3	71.7	78.8	66.5	68.4	40.2	59.5
	7B	300B	35.0	77.6	80.0	69.9	74.4	45.9	63.8
Griffin	1B	300B	29.5	67.2	77.4	65.2	67.0	36.9	57.2
	3B	300B	32.6	73.5	78.1	67.2	71.5	41.4	60.7
	7B	300B	39.3	78.6	81.0	72.6	75.4	47.9	65.8
	14B	300B	49.5	**81.4**	**81.8**	**74.1**	**79.1**	**50.8**	**69.5**

These models can sometimes match Transformer perplexity and standard academic benchmark results

But do they have the same recall and in-context learning abilities expected of Transformers?

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Agenda

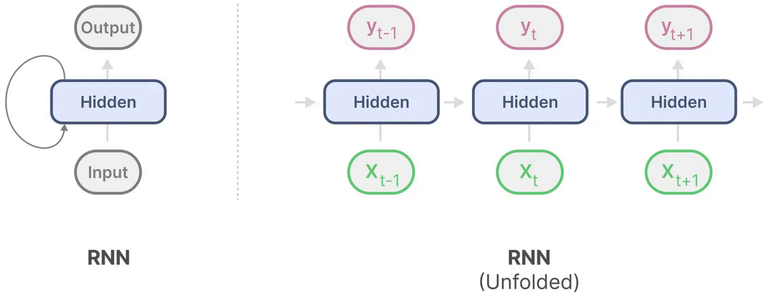
- Introduction, motivation, prior approaches
- Linear state space models (SSMs) overview
- S4, convolutions, parameterization
- S5, diagonalization, parallel scans
- S6/Mamba/Hawk/Griffin, data-dependent dynamics
- **Challenges and opportunities**

Recall challenges remain

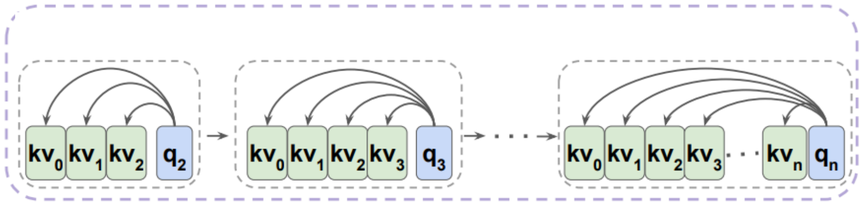
Many Mamba for X papers quickly followed showing strong results in vision, diffusion etc., suggesting these LTV systems can be very strong models.

But recall/copying problem in language seems to persist…:

- **Repeat after me: Transformers are better than state space models at copying https://arxiv.org/abs/2402.01032**
- **Simple linear attention models balance recall-throughput tradeoff https://arxiv.org/abs/2402.18668**
- **Can Mamba learn how to learn? A comparative study on in-context learning tasks: https://arxiv.org/abs/2402.04248**



Vs.

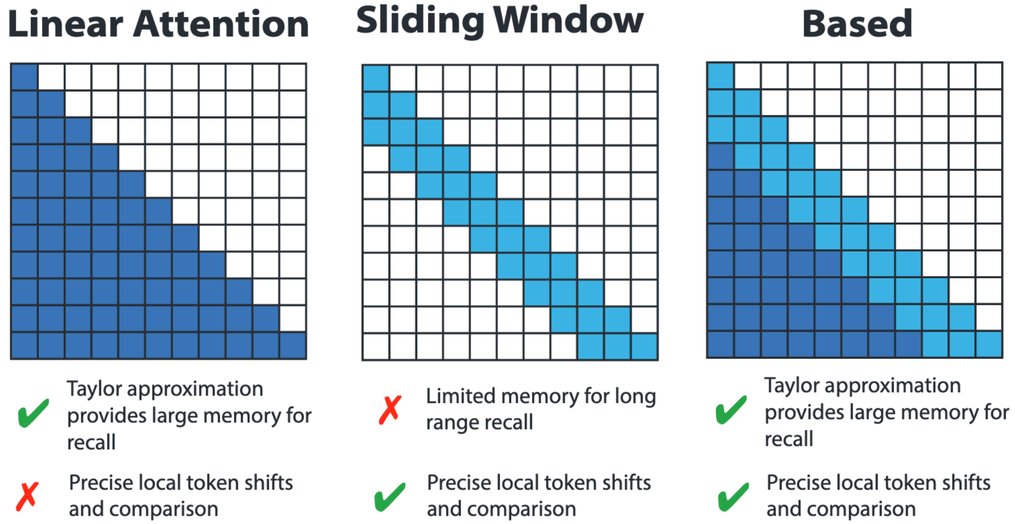


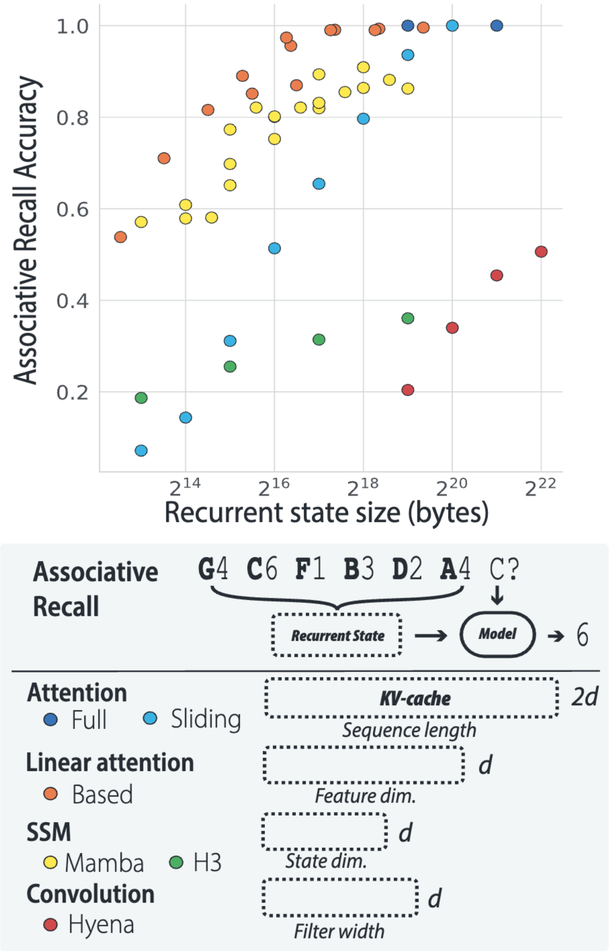
Recall challenges remain

Griffin, Section 6.2 Copying and Retrieval capabilities:

From Figure 6(c), we see that while ***Hawk*** *can do reasonably well on the task for very short phonebook lengths, it* ***fails to memorize and retrieve*** *the correct phone number when the phonebook length grows, similar to the observation made by Jelassi et al. (2024)…Our* ***Transformer baseline*** *can* ***almost perfectly solve*** *this task up to the training sequence length…Interestingly,* ***Griffin*** *can perfectly solve this task up to a context length that matches its local attention* ***window size of 1024****… Once the context length is long enough such that the local attention window* ***does not cover*** *the whole phonebook,* ***performance starts to degrade****…While the performance of Griffin is promising for the ability of models with fixed-size state to solve copying and retrieval tasks,* ***our results suggest more work is needed to improve these capabilities for such models.***

More sliding window hybrids





Arora et al. Simple linear attention language models balance the recall-throughput tradeoff. 2024

Possible Solutions?

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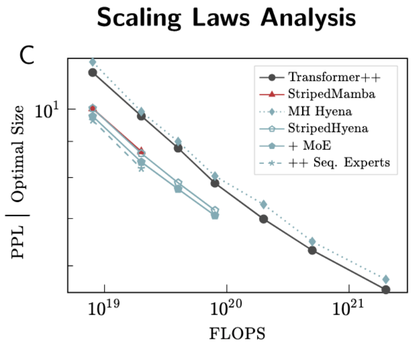
- **Training flop/cost penalty: Can more tokens (or better data) help fixed state size methods make better use of state?**



Poli et al 2024. Mechanistic Design and Scaling of Hybrid architectures

Poli et al. Mechanistic Design and Scaling of Hybrid Architectures. 2024

Possible Solutions?

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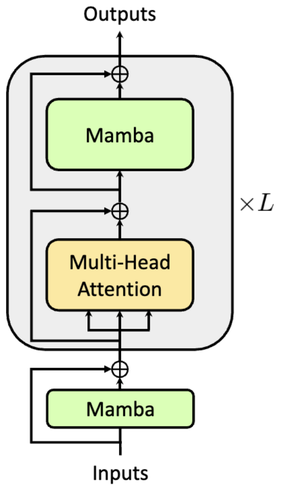
- **Hybrid models?**



Poli et al. Mechanistic Design and Scaling of Hybrid Architectures. 2024
Park et al. Can Mamba Learn How to Learn? A Comparative Study on In-Context Learning Tasks. 2024

Possible Solutions?

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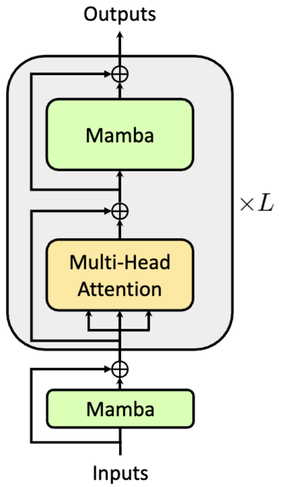


- **Better dynamic storage/retrieval mechanisms?**

Taxonomy

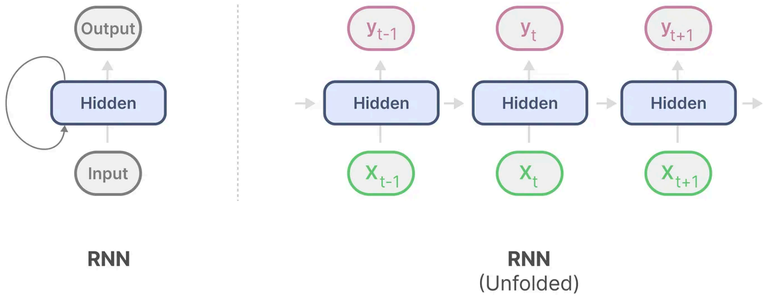
Fixed state size methods

Time-invariant
- LTI SSMs/RNNs (e.g. S4/S5/LRU)
- Linear Attention (e.g. Retnet)
- Long Convolutions (*Distilled)

Time-varying
- Nonlinear RNNs
- LTV SSMs/RNNs (e.g. Mamba/S6/Hawk/Griffin)
- Gated Linear Attention

Dynamic state size methods

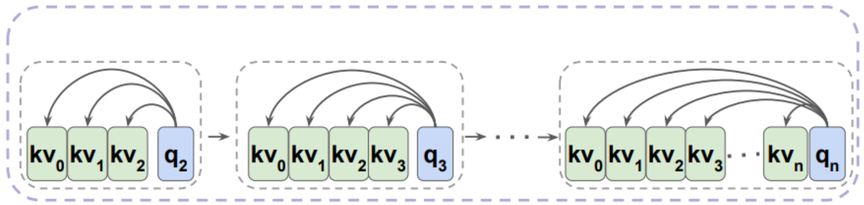
- Softmax attention
- Other methods whose state grows with sequence



RNN

RNN
(Unfolded)

Vs.



Thank you!