Diffusion capacity of single and interconnected networks

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ACDL 2023

Single Networks

Large, Massive and Complex (social networks, telecommunication networks, smart grid, biological networks, financial networks, etc)

Handbook of Massive Data Sets (Springer 2013)

(James Abello, Panos M. Pardalos, Mauricio G.C. Resende)

Networks of Networks

- IoT (**Internet of Things**)

A worldwide network of interconnected objects that are uniquely addressable via standard communication protocols.

- Cooperative networks
- Multicast networks
- Interdepended networks
- Sustainable interdepended networks

New Challenges

Schieber, T., Carpi, L., Díaz-Guilera, A., Pardalos, P.M. *et al.*

Quantification of network structural dissimilarities.

Nat Commun **8**, 13928 (2017). https://doi.org/10.1038/ncomms13928

Carpi, L.C., Schieber, T.A., Pardalos, P.M. *et al.*

Assessing diversity in multiplex networks. *Sci Rep* **9**, 4511 (2019). https://doi.org/10.1038/s41598-019-38869-0

Summary of new work

Understanding of diffusive and spreading processes in networks remains challenging when dynamics of the network is complex. Here, we propose a quantity to reflect the potential of a network node to diffuse information, that may serve to develop interventions for improved network efficiency.

Natural and artificial diffusive processes from the most varied contexts are omnipresent in our everyday lives

- Diffusion magnetic resonance
- Infectious agents that attack our immune system
- Billions of individuals commuting daily between different geographical regions constitute the highly complex global human mobility system
- Gossip spreads through vast complex social networks
- Many others...

Diffusion in Networks

- Structural and dynamical properties of diffusion processes have been successfully modeled by networks, structures that are able to encompass this complex combination.

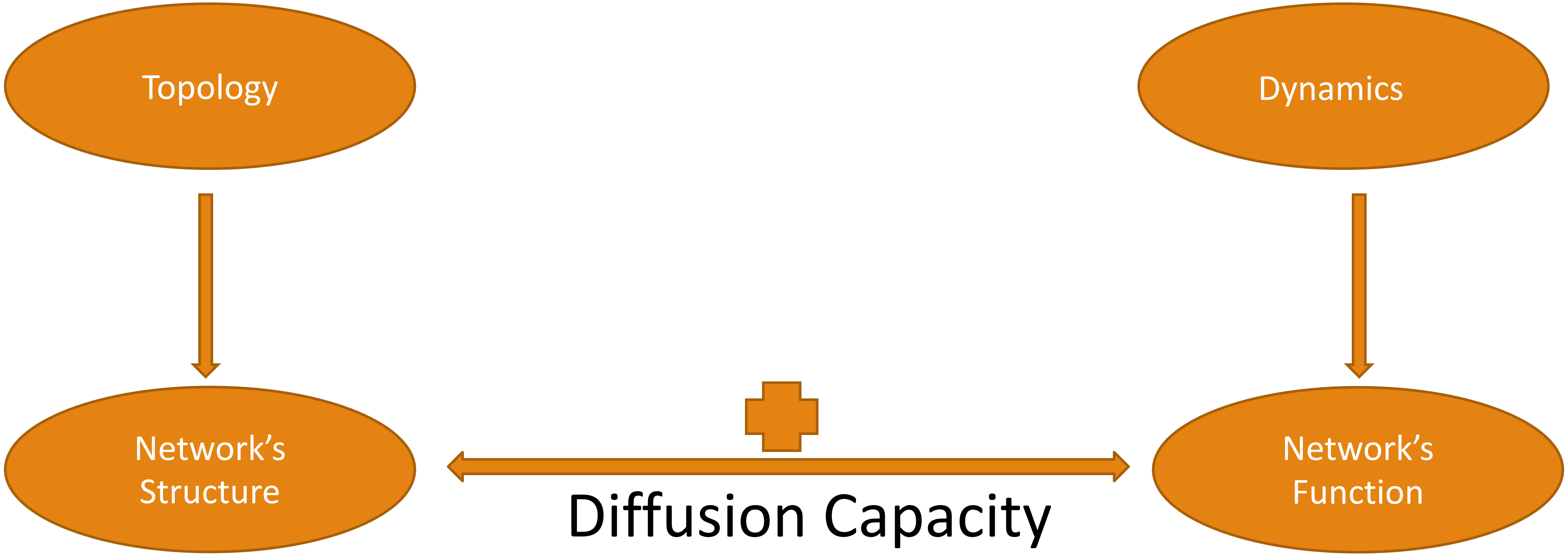
- The influence of specific topological features, such as community structures or degree heterogeneity of nodes on diffusion processes, has been described in several works.

- One interesting point that still needs to be addressed in the literature is the study of the way individual nodes change their performance as diffusing agents during the evolution of the diffusive process.

Diffusion Capacity

Considering that nodes change the way they diffuse as the process evolves, we propose a new measure called Diffusion Capacity that is able to track the evolution of the node's performance. Diffusion-Capacity quantifies the evolving diffusive ability of nodes through the use of a weighted distance distribution that allows the inclusion of dynamical features of the process. We introduce in this work a method that brings many possibilities for strategic interventions to design more efficient diffusive structures.

Diffusion Capacity



Climate Networks

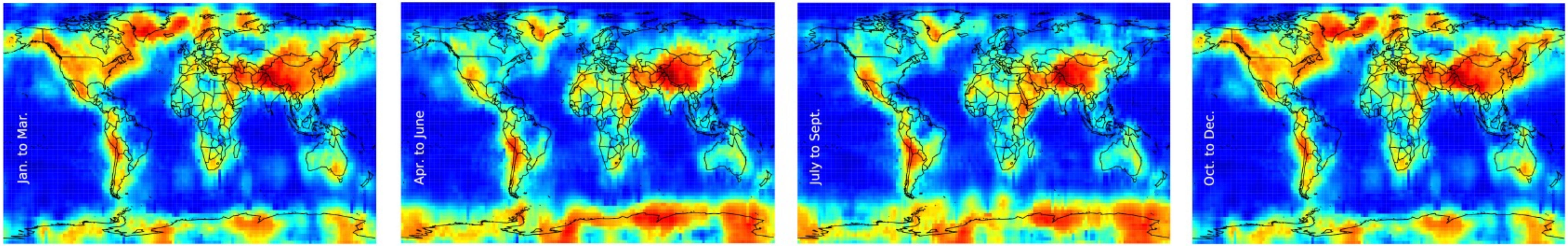
As a real world example of heat diffusion we use the gridded reanalysis dataset of Surface Air Temperature (SAT):

Kalnay, E. The ncep/ncar 40-year reanalysis project. Bull. Am. Meteorolog. Soc. 77, 437–472 (1996).

- Topology: gridded Earth network
- Dynamics: heat flow through the Earth surface

Climate Networks

Colors of geographical points correspond to the diffusion-capacity mean value of each season computed through daily surface air temperature data.



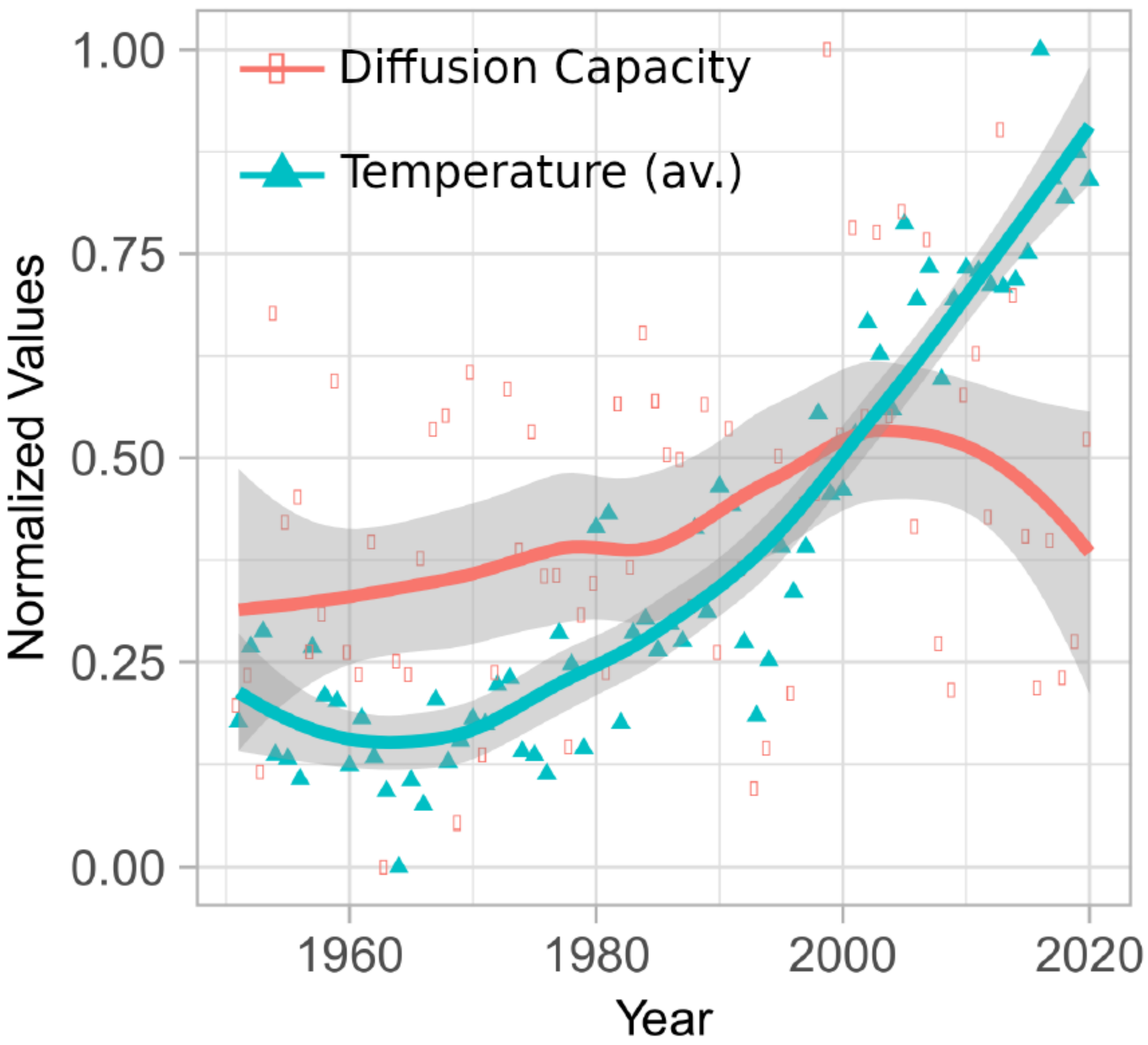
Climate Networks

Some interesting global features can be observed from the diffusion-capacity spatial patterns.

- In general, superficial air over oceans show lower diffusion-capacity values than superficial air over land, increasing gradually in coastal waters.
- Considering only land areas, higher diffusion-capacity values are observed in places of higher altitudes as well as in winters of colder places
- Another interesting fact is that diffusion-capacity variation among seasons is much lower between tropics than in extratropical regions.

Climate Networks

Annual values of diffusion Capacity (orange) and annual temperature values (green) for the global surface air temperature (SAT) climate network. It is possible to see that while temperatures rise constantly, diffusion-capacity drastically changes its tendency around 2000. It is worth mentioning, that extreme weather events have increased significantly in the last 20 years almost duplicating the number of natural disasters



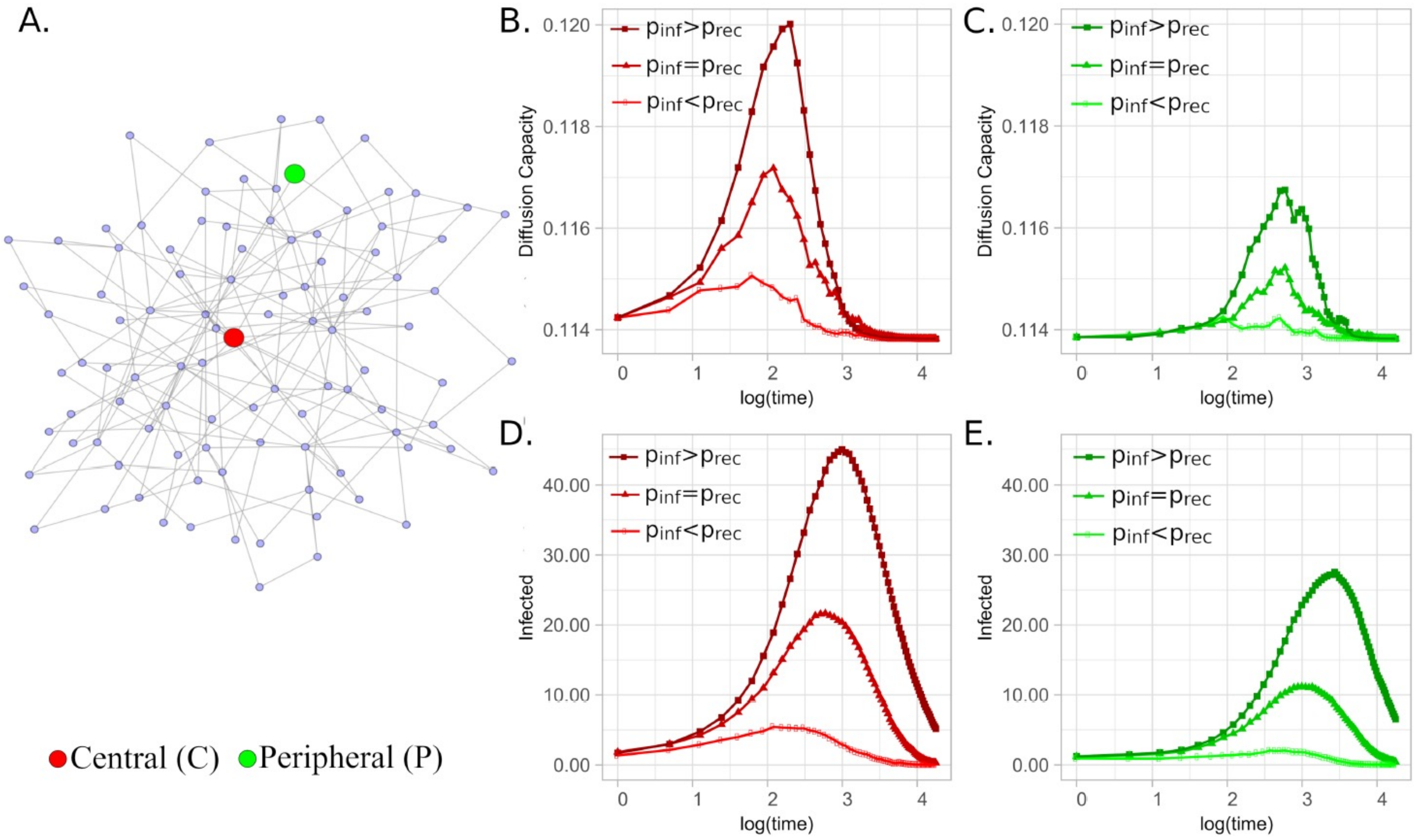
Disease Spreading

The SIR (succeptible-infections-removed) model is a simple mathematical model for the spread of epidemic diseases, in which, nodes can be susceptible, infected, or recovered from an infectious agent. Recovered nodes are immune to the disease, while a susceptible node can become infected if it is in contact with an infected node.

- We consider the evolution of the SIR model in a small network when one central node is initially infected, and also when a peripheral node initiates the epidemic process. In this experiment, we consider the probability of a susceptible node to become infected is $p_{inf} = 0.1$ and different recovery rates p_{rec} values.

Disease Spreading

Small network highlighting a central (red) and a peripheral (green) node (A). Network diffusion-capacity evolution of a SIR model when the central node (B) and a peripheral node (C) is initially infected, for three different infection probabilities. D, E depict the evolution of the number of infections respectively. All figures show mean values of 100 realizations considering a constant $p_{inf} = 0.1$ and $p_{rec} = 0.05, 0.10\ and\ 0.20$.



Disease Spreading

These figures show that the initial diffusion capacity is **higher** when the process is initiated in the **central** node. As the process evolves, the diffusion capacity increases until a maximum that appears **earlier** than the maximum number of cases, showing itself as an early indicator of the peak of the epidemic process.

Interconnected Networks

- Structures that represent a system composed by several networks interacting simultaneously.

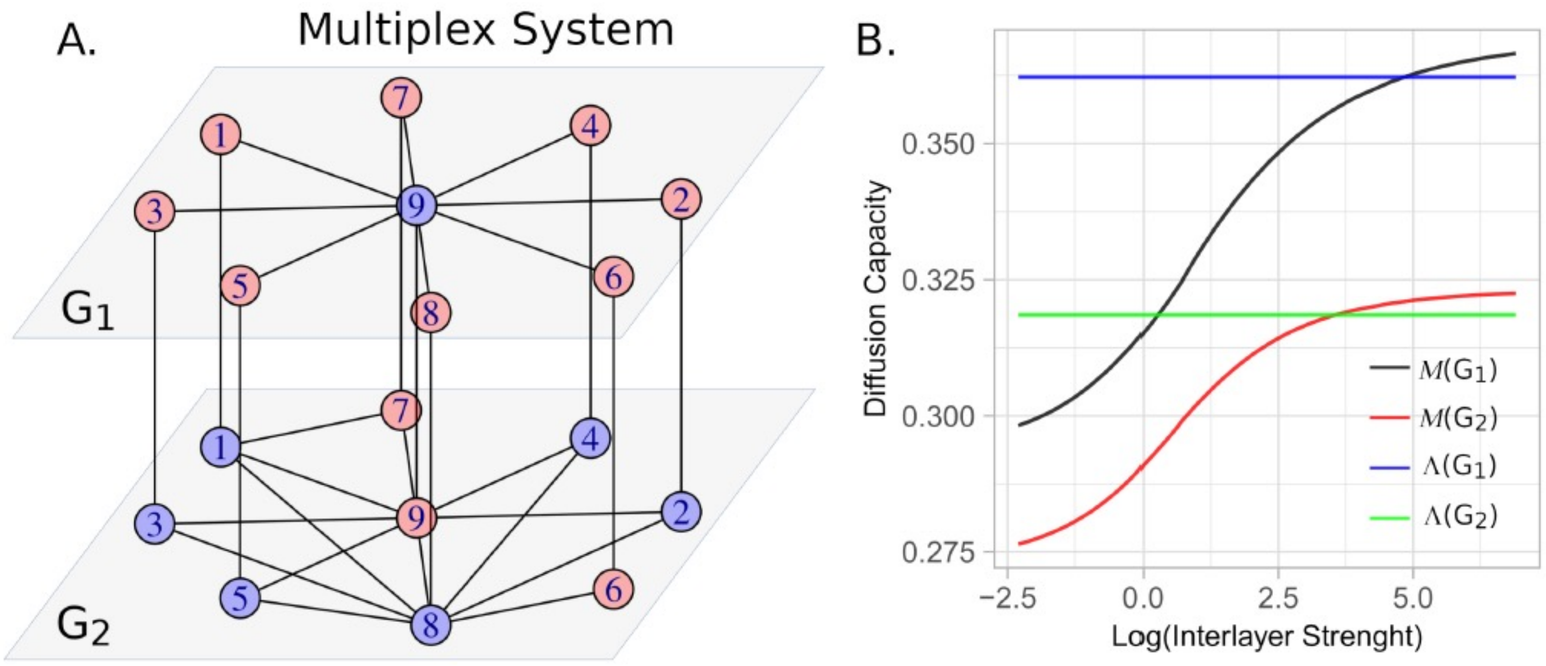
- These systems are a good representation of natural systems since they are the result of the complex interaction of many subsystems.

- Considering this we propose a measure called **Relative Gain (G)** that compares the diffusive performance of an element acting in an isolated network and its performance when it is part of an interconnected structure.

Multiplex System

Small multiplex system coupled by an interlayer strength D_x.

In these structures, if there are no interlayer connections, diffusion occurs independently in each layer. However, when the interlayer strength is low, the diffusion time may become excessively long, as these weak interlayer connections slow down the dynamics of both layers. On the other hand, a strong interaction between layers enhances diffusion.



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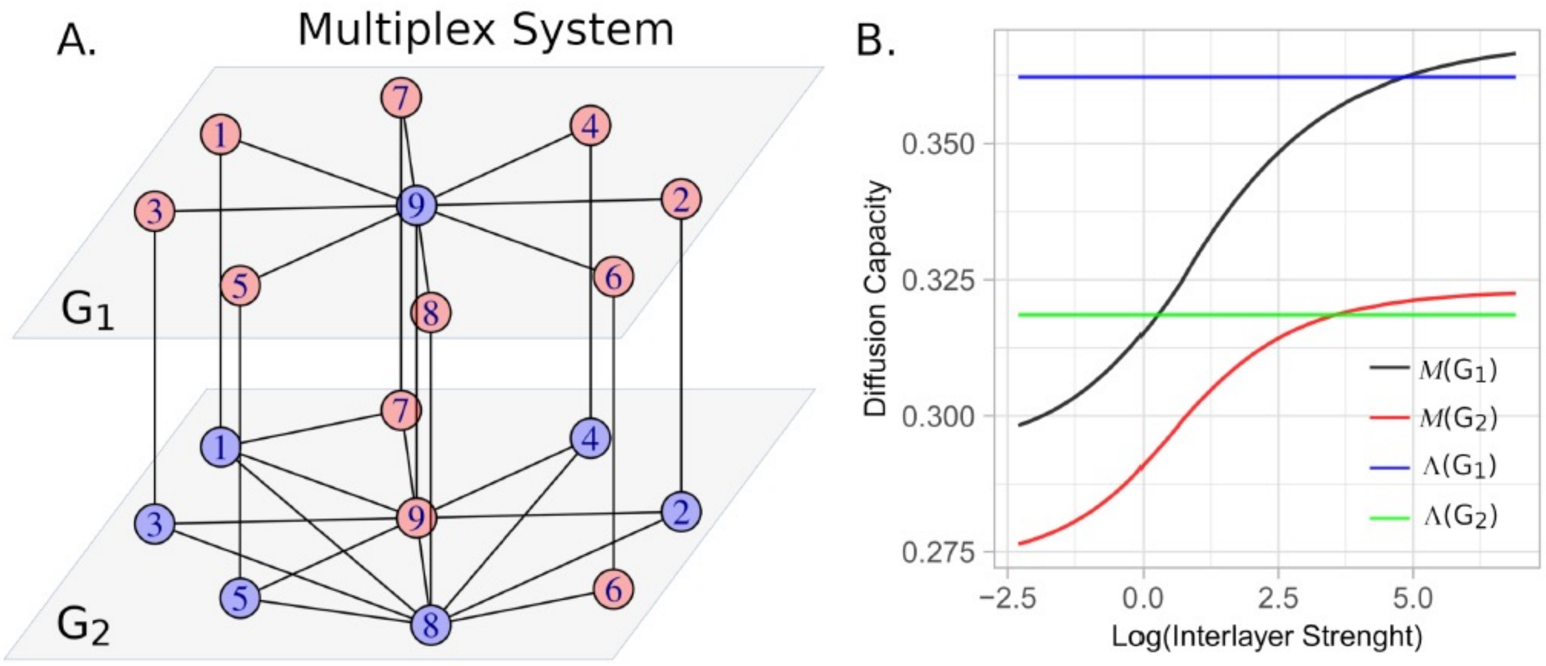


A. Blue nodes: decrease the diffusion capacity when multilayered. Red Nodes: increase the diffusion capacity when multilayered. B. Evolution of multilayered diffusion capacity, M, When increasing the interlayer strength. The Λ values represent diffusion capacities when the networks are in isolation.

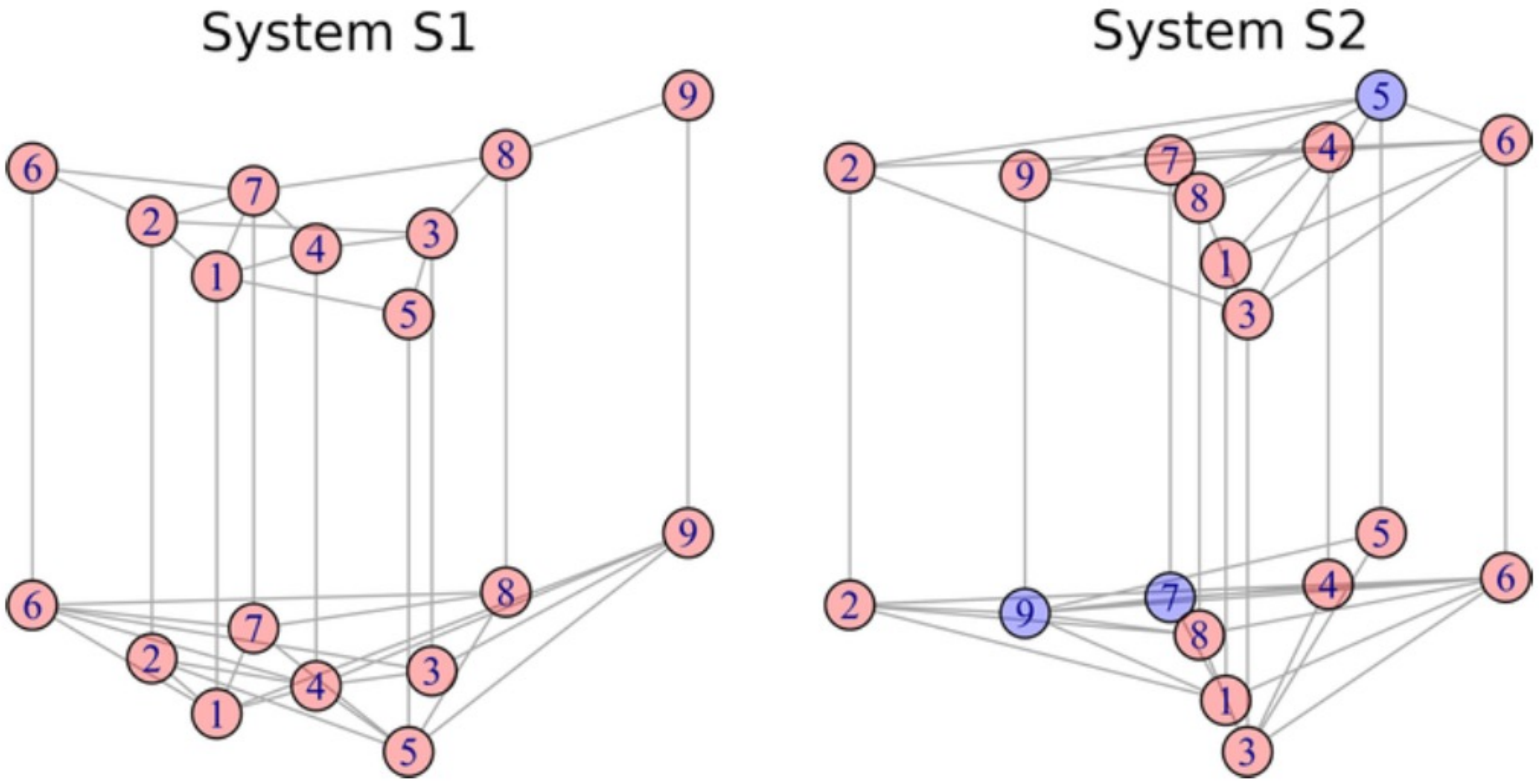
Relative Gain (G)

The ratio between the multilayer diffusion capacity and the diffusion capacity of the isolated layer.

- Relative gain is also an important concept for the study of superdiffusion, phenomenon in which diffusion processes reach a steady state faster on a multilayer structure than in any of their constitutive layers in isolation.

We use 1000 different random initial conditions and we observe the number of times superdiffusion is developed. Results reveal that superdiffusion is found in 99.8% of cases for S1 and in 75.6% of the cases in S2, showing the strong influence of the network topology and its dependence on initial conditions.

It is interesting to note that, considering the approach based on spectral properties of Laplacian matrices, both structures are considered superdiffusive. This experiment highlight the advantage of the possibility of quantifying DC as it allows the design of structures with specific diffusive requirements and also to investigate the initial conditions in which the structure becomes superdiffusive.

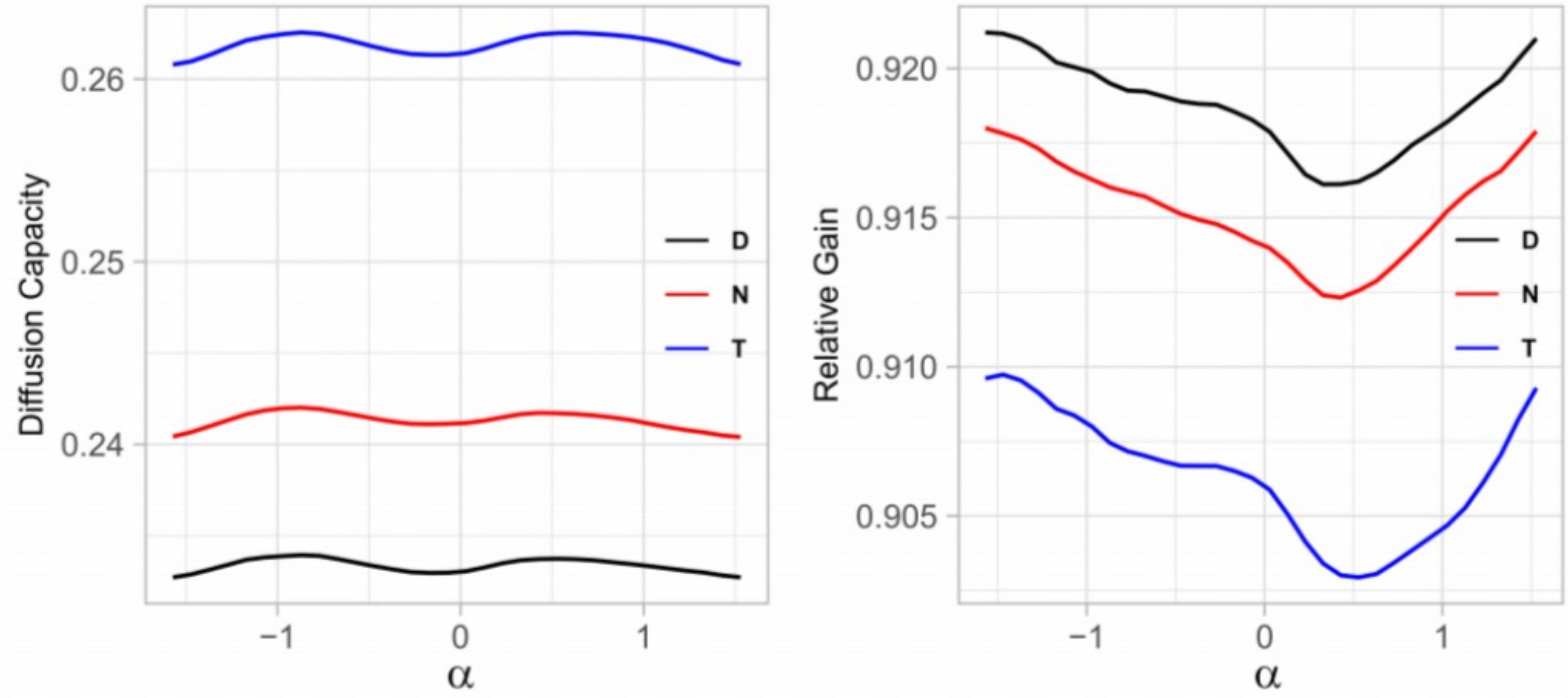


Diffusion Capacity in a multilayer brain network

We study the involvement of the central nervous system via electroencephalography (EEG) of the 9-layer network constructed by Huang et al., considering the three types of patients: complete deafness (D), tinnitus (T), and normal controls (N).

- Huang, L., Wang, C.-D. & Chao, H.-Y. Hm-modularity: A harmonic motif modularity approach for multi-layer network community detection. IEEE Transactions on Knowledge and Data Engineering 33, 2520–2533 (2021)

Interesting to note that the Tinnitus system has a higher diffusion capacity than the other ones but, on the other hand, a lower relative gain in the multilayered system. The opposite is true for patients with complete deafness.



Methods

Weighted Node Distance Distribution

Let $D_g(i,j)$ and $D_w(i,j)$ the geodesic and weighted distance (the minimum sum of the inverse of the weights connecting both nodes) between vertices i and j, respectively.

$$\Delta_{i,j} = D_w(i,j)/D_g(i,j)$$

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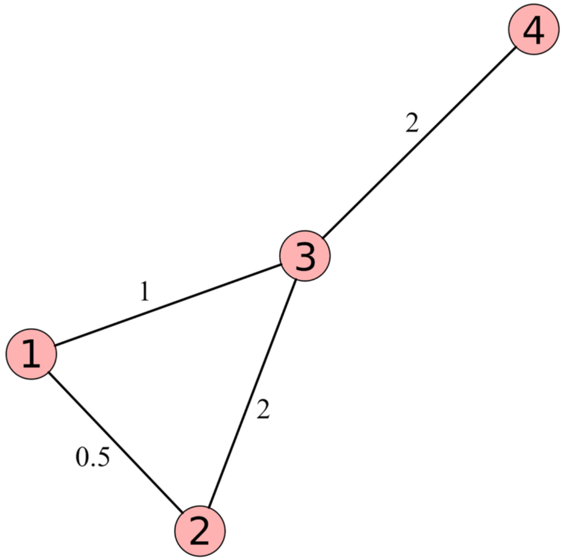

$$\Delta_{i,j} = D_w(i,j)/D_g(i,j)$$

$$\Delta_{1,2} = 1.5/1$$

$$\Delta_{1,3} = 1/1$$

$$\Delta_{1,4} = 1.5/2$$

Methods

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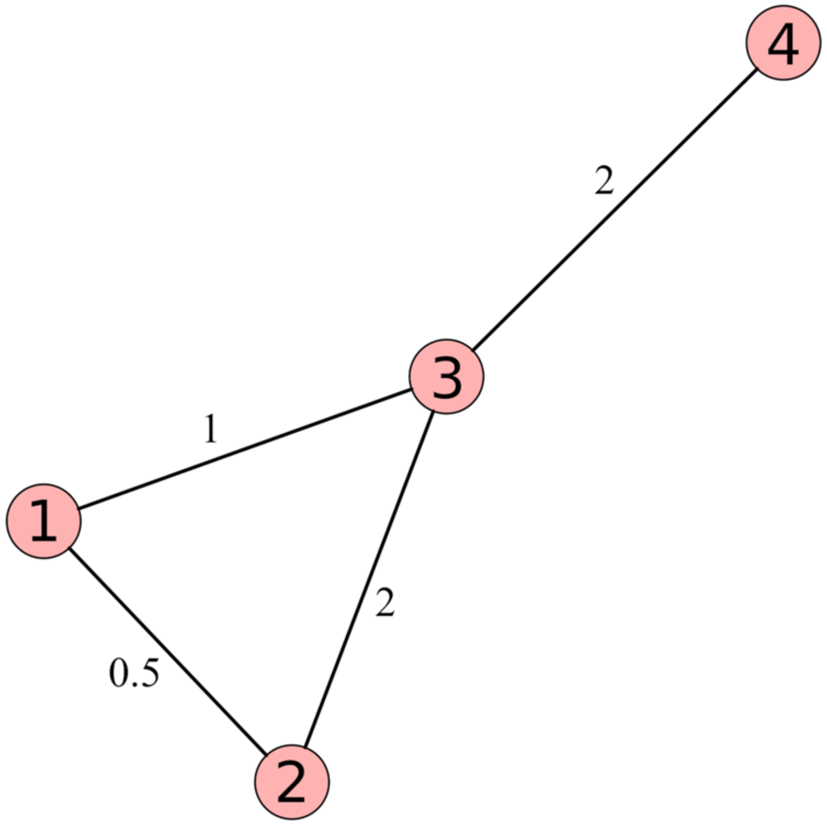
$$\Delta_{i,j} = D_w(i,j)/D_g(i,j)$$

For $d = 1, 2 \ldots, |V| - 1, \infty$, let $\Gamma_i(d) = \{y \in V \mid D_g(i,j) = d\}$. We define:

$$[p_i^+(d), p_i^0(d), p_i^-(d)] = \frac{1}{|V|-1} \sum_{y \in \Gamma_i(d)} [\max(1 - \Delta_{i,j}, 0), \min\left(\Delta_{i,j}, \frac{1}{\Delta_{i,j}}\right), \max(1 - \frac{1}{\Delta_{i,j}}, 0)]$$

Methods



$\Gamma_{1}(1) = \{ y \in V \mid D_g(i, j) = 1 \} = \{2,3\}$

$\Gamma_2(1) = \{ y \in V \mid D_g(i, j) = 2 \} = \{4\}$

$3\ p_1^+(1) = \max(1 - \Delta_{1,2}, 0) + \max(1 - \Delta_{1,3}, 0) = 0$

$3\ p_1^0(1) = \min\left(\Delta_{1,2}, \frac{1}{\Delta_{1,2}}\right) + \min\left(\Delta_{1,3}, \frac{1}{\Delta_{1,3}}\right) = \frac{1}{1.5} + 1$

$3\ p_1^-(1) = \max\left(1 - \frac{1}{\Delta_{1,2}}, 0\right) + \max\left(1 - \frac{1}{\Delta_{1,3}}, 0\right) = 1 - 1/1.5$

$3\ p_1^+(2) = \max(1 - \Delta_{1,4}, 0) = 1 - 1.5/2$

$3\ p_1^0(2) = \min\left(\Delta_{1,4}, \frac{1}{\Delta_{1,4}}\right) = 1.5/2$

$3\ p_1^-(2) = \max\left(1 - \frac{1}{\Delta_{1,4}}, 0\right) = 0$

Methods

Weighted Node Distance Distribution

The weighted node distance distribution wNDD is then defined for each node, as:

$$\boldsymbol{P}_i = [p_i^+(1), p_i^0(1), p_i^-(1), \dots, p_i^+(|V|-1),\ p_i^0(|V|-1), p_i^-(|V|-1), p_i(\infty)]$$

Methods

Diffusion Capacity

The wNDD equals $P_{ref} = (1,0,0,\dots,0)$ is the form of the distance distribution (wNDD) of a node in a fully connected network with weights tending to infinite.

We define the Diffusion Capacity of a node i as the inverse of the cumulative Jensen-Shannon divergence between the wNDD and P_{ref}.

$$\boldsymbol{\Lambda_i(G) = \left[CDD\left(P_i, P_{ref}\right)\right]^{-1}}$$

Methods – Multilayer Networks

Paths in multilayer networks

Use the concepts of single and doubly connected nodes proposed in the Lace Expansion method in the following way:

- $D_w(i,j)$ is the shortest distance over all paths connecting i and j in the multilayered system, being $\boldsymbol{P}_i$ its corresponding probability distribution;
- $D_w^\beta(i,j)$ the shortest distance over all paths connecting i and j such that at least one node in the path belongs to a different layer, G_β with $\beta \neq L$, being $\boldsymbol{P}_i^\beta$ its corresponding probability distribution.

Methods – Multilayer Networks

The Node Diffusion Capacity ($\boldsymbol{M}_i$) for each vertes i in layer L, for all $L = 1,2, \dots, M$ is defined as:

$$\left[\frac{1}{2}\,\mathrm{CDD}(\mathbf{P}_{\mathrm{i}}, \mathbf{P}_{\mathrm{ref}}) + \frac{1}{2M-2}\sum_{\beta=1,\beta\neq L}^{M} CDD\left(\boldsymbol{P}_i^{\beta}, \boldsymbol{P}_{ref}\right)\right]^{-1}$$

Relative Gain

To explore this relationship between single and multilayer diffusion capacities, we define the relative gain G as the ratio between the multilayer diffusion capacity and the diffusion capacity of the isolated layer.

$$G_i = M_i / \Lambda_i$$

Discussion

In conclusion, our work introduces an approach to quantify the Diffusion Capacity of nodes in a network and identify potential interventions for improving the system's efficiency. We believe that our approach has broad applications in various fields, including transportation, social networks, and epidemiology, among others.

References

Schieber, T.A., Carpi, L.C., Pardalos, P.M. *et al.* Diffusion capacity of single and interconnected networks. *Nat Commun* 14, 2217 (2023). https://doi.org/10.1038/s41467-023-37323-0 (see also supplementary information)

Schieber, T., Carpi, L., Díaz-Guilera, A. *et al.* Quantification of network structural dissimilarities. *Nat Commun* **8**, 13928 (2017). https://doi.org/10.1038/ncomms13928