Value of Information in Neural Networks

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Value of Hartley's information

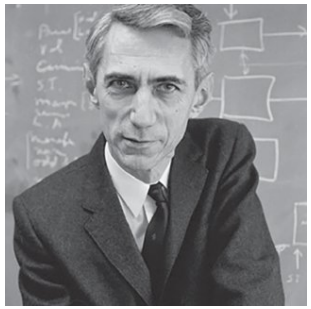
Feed-forward neural networks and VoI

Forecast and VoI

Values of Boltzmann's and Shannon's information

Solutions of VoI

Information and its Value



Claude Shannon

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(Shannon, 1948)

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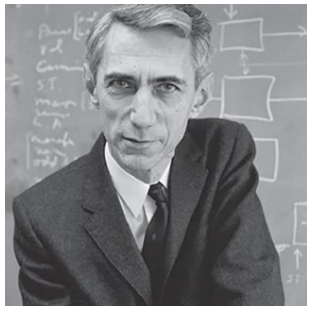

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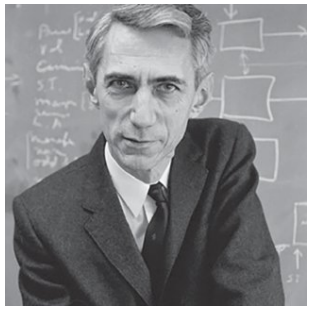

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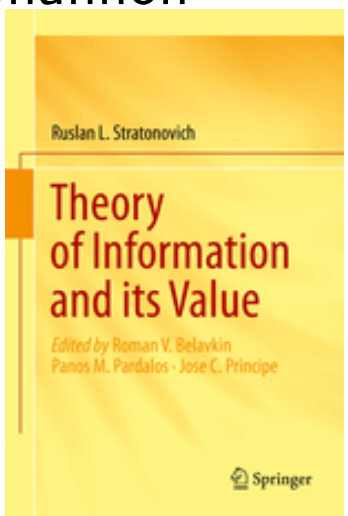

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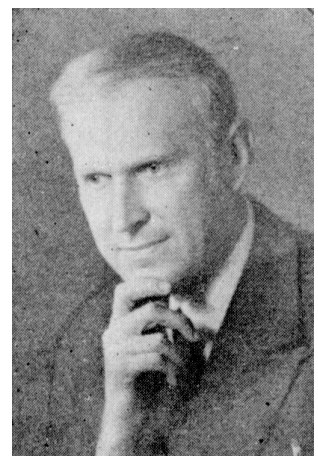

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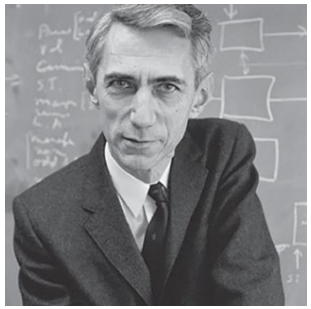

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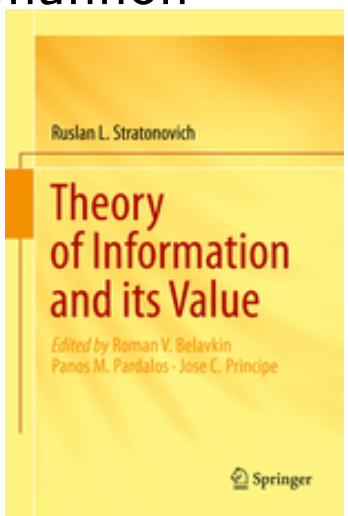

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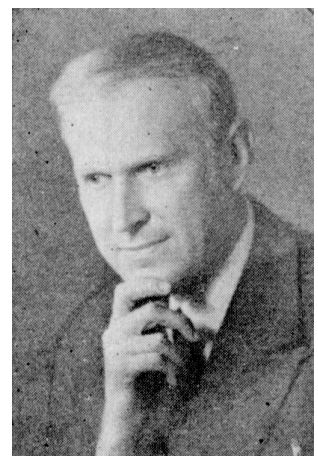

Ruslan Stratonovich

Optimal algorithms for pattern recognition (Stratonovich, 1968)

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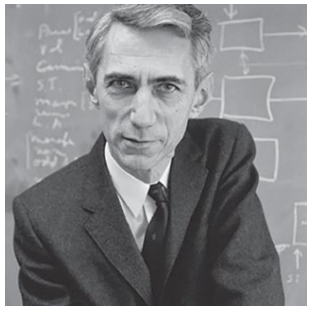

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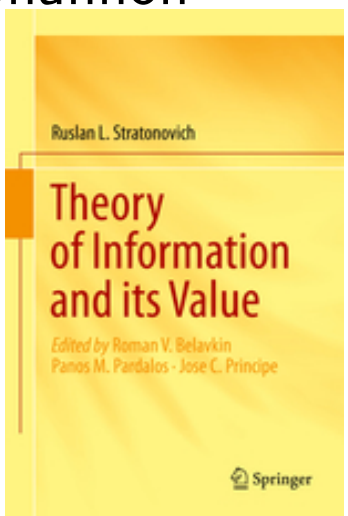

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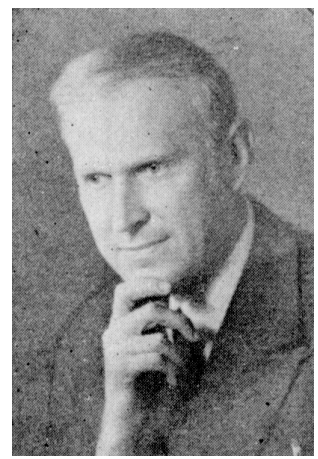

Ruslan Stratonovich

Effectiveness of statistical methods in problems of synthesis of algorithms for function approximation (Stratonovich, 1969)

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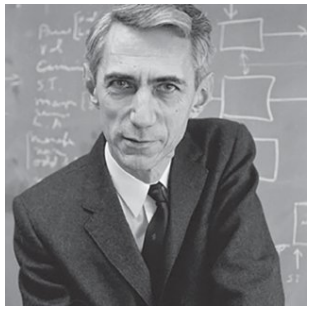

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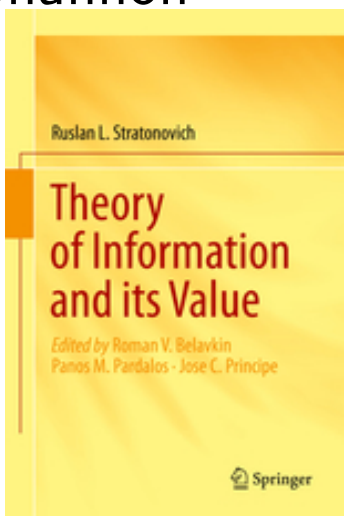

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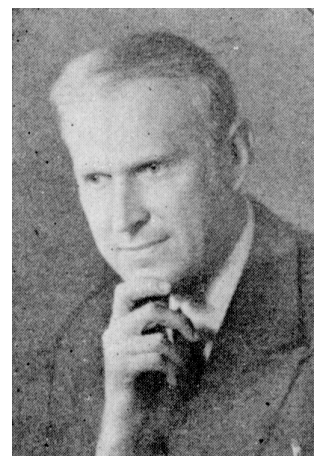

Ruslan Stratonovich

Optimal expansion of functional subspace in algorithms for function and probability density approximation (Stratonovich, 1970b)

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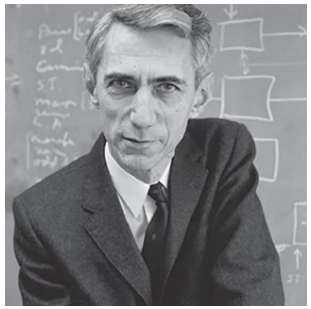

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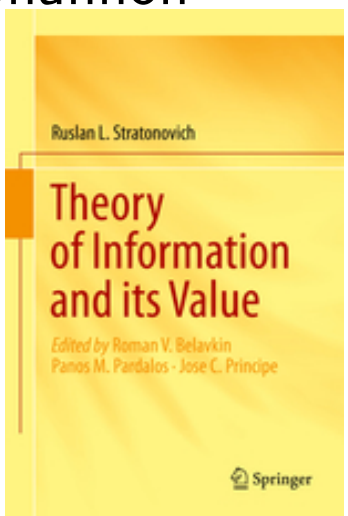

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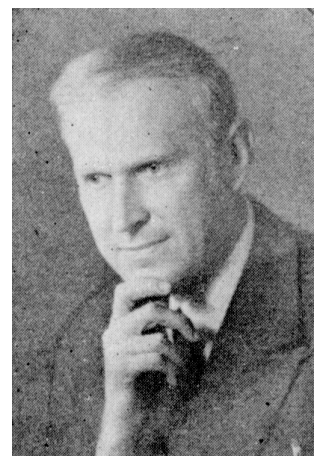
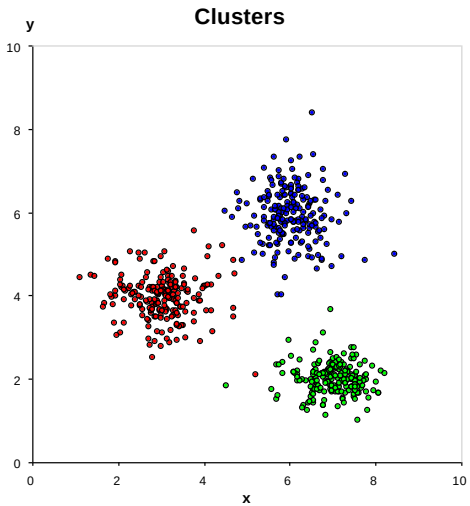


Ruslan Stratonovich

Canonical recursive system of equations for optimal adaptive algorithms (Stratonovich, 1970a)

Information and Quality of Decisions



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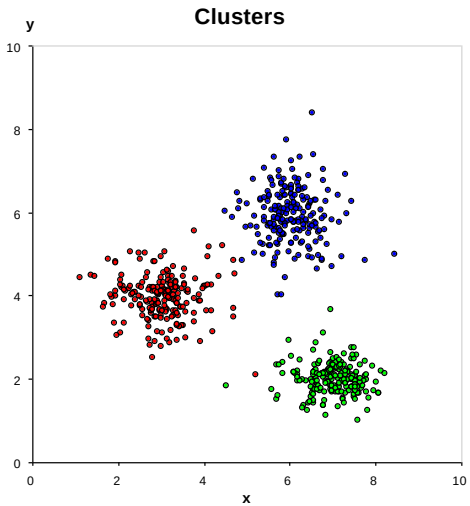

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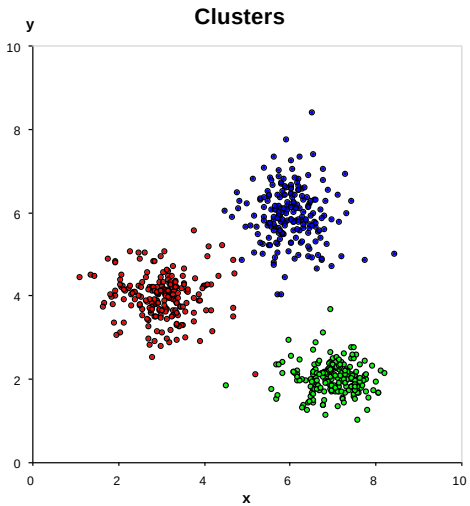

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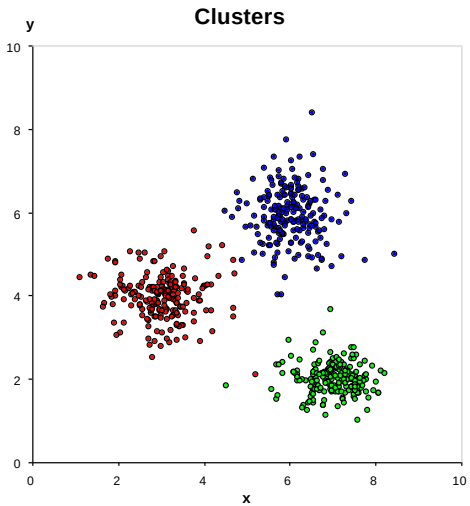

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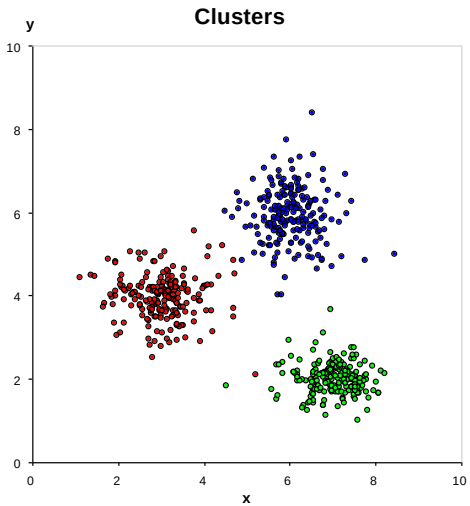

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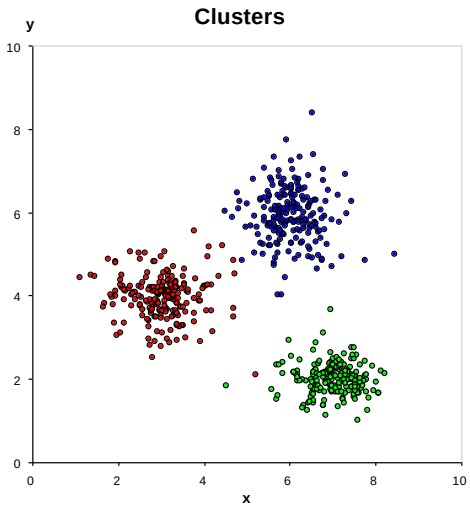

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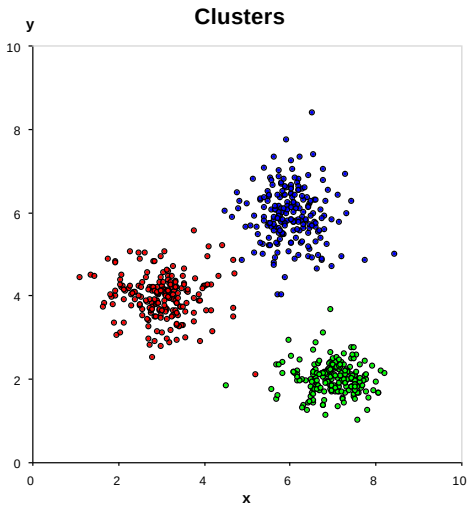

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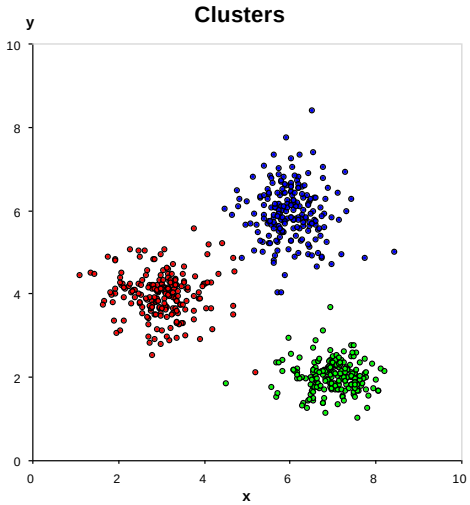

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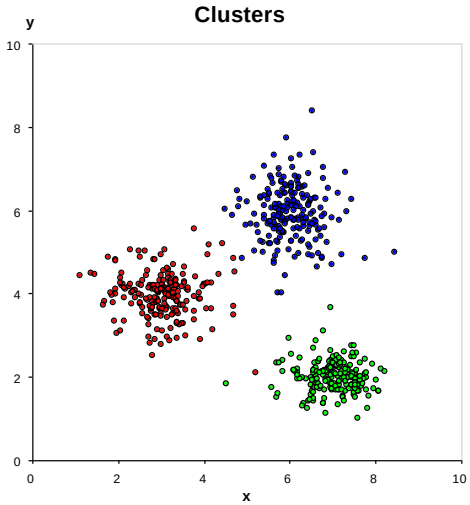

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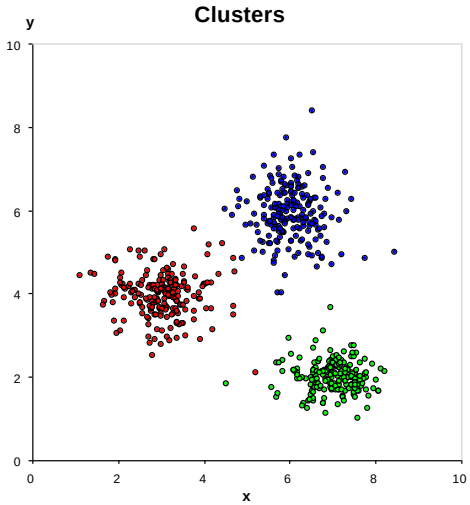

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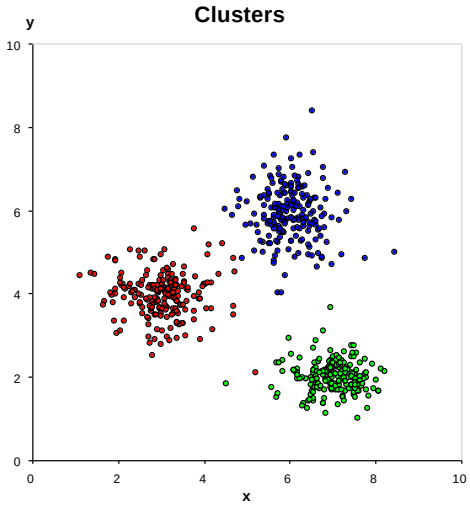

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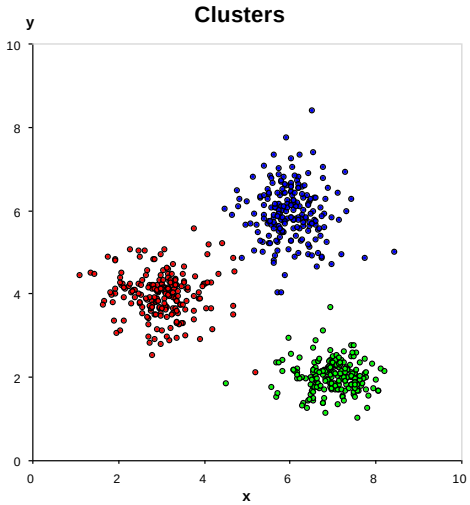

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- The *value of Harley information* (Stratonovich, 1965):

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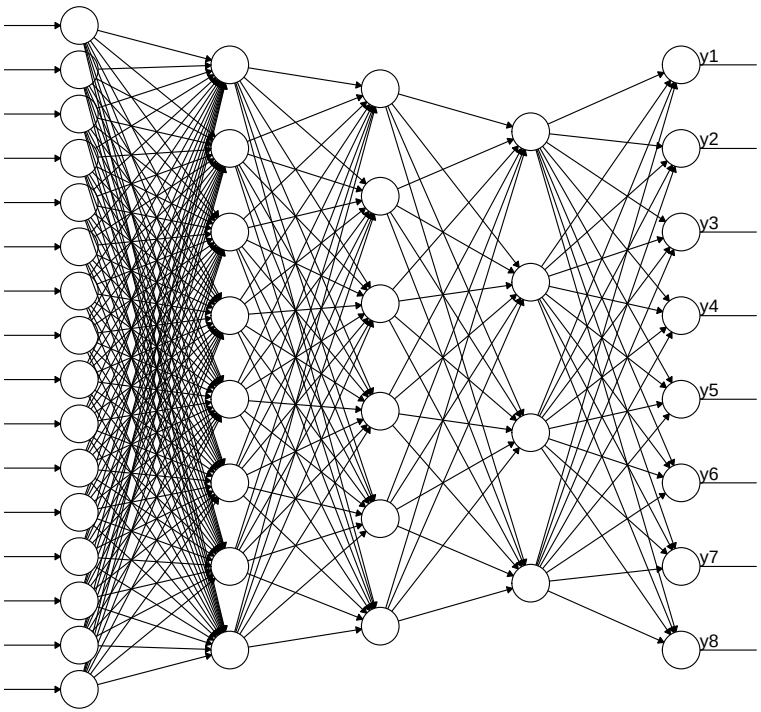
Remark

We are looking for optimal function $z(x) = z \circ y(x)$ *subject to* $|Z| \leq k$.

Feed-forward neural networks



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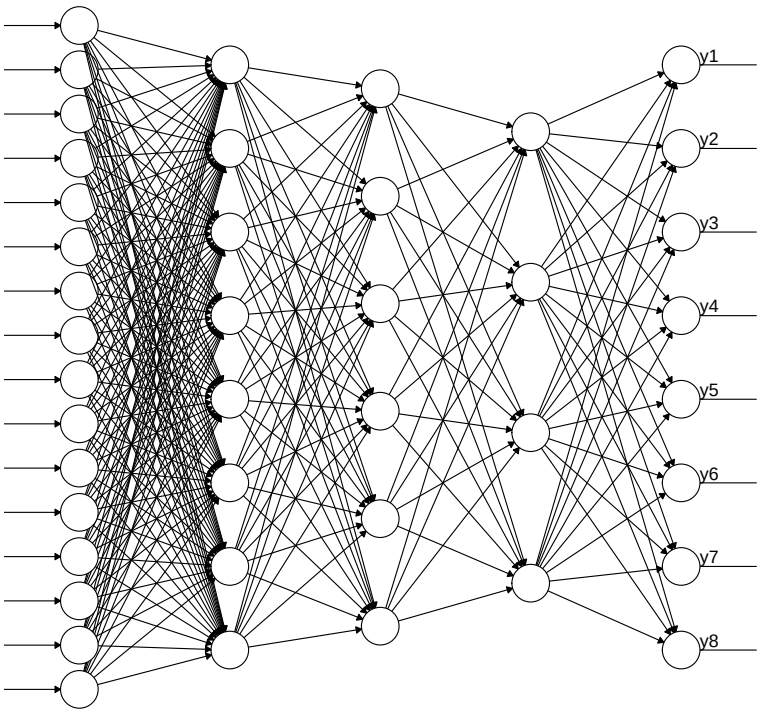

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1. Initialise the weights w_{ij} (e.g. at random).
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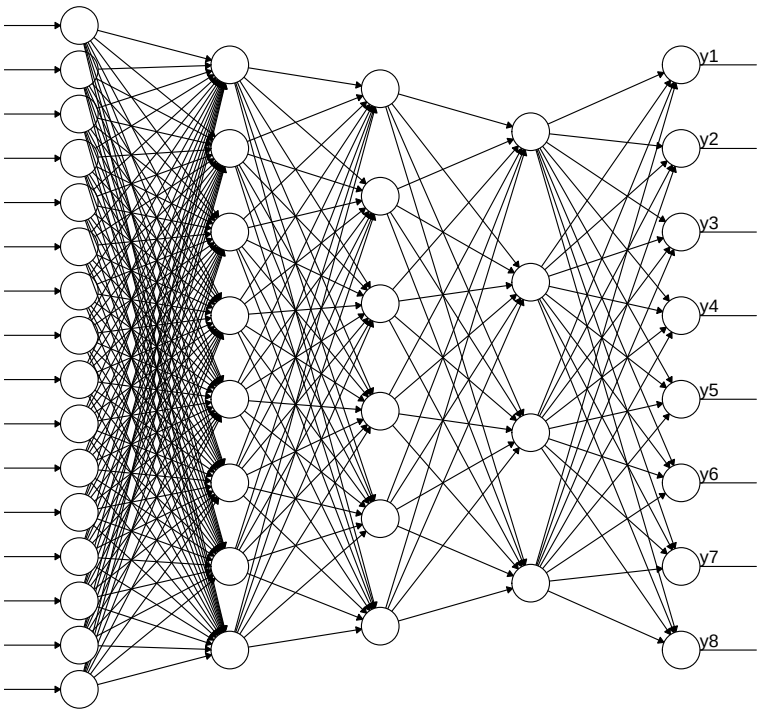

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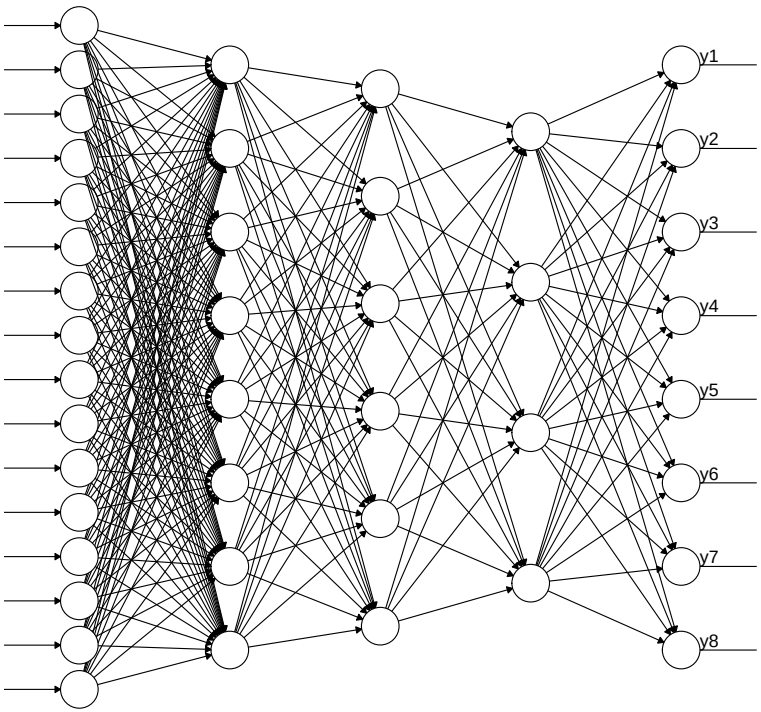

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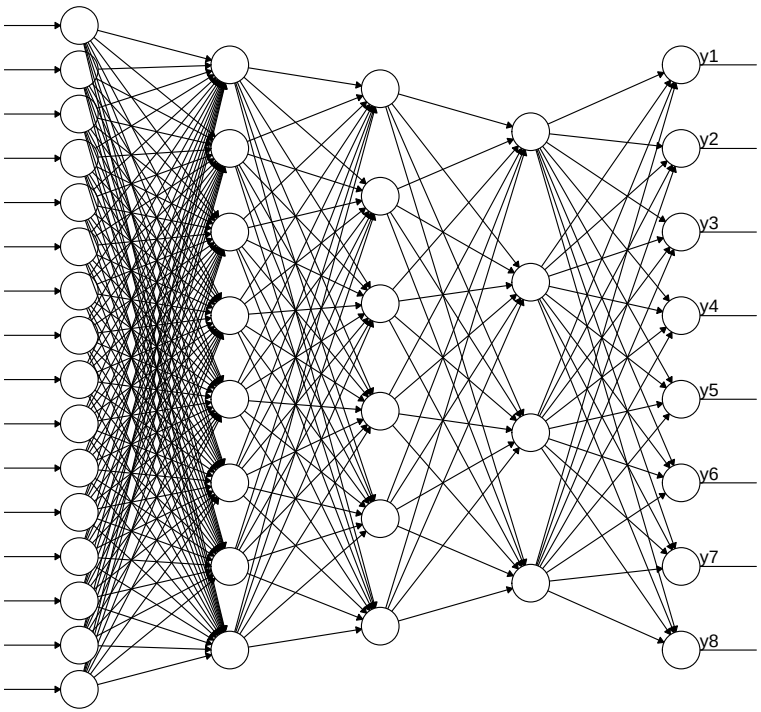

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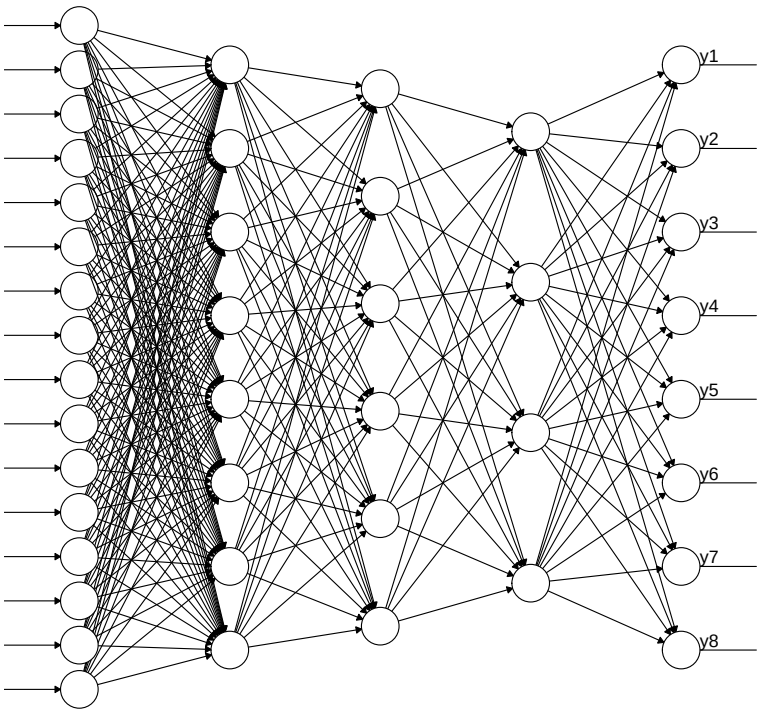

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2. Repeat
   1. Feed the network with an input y from a training set.
   2. Compute the network's output $z(y)$.
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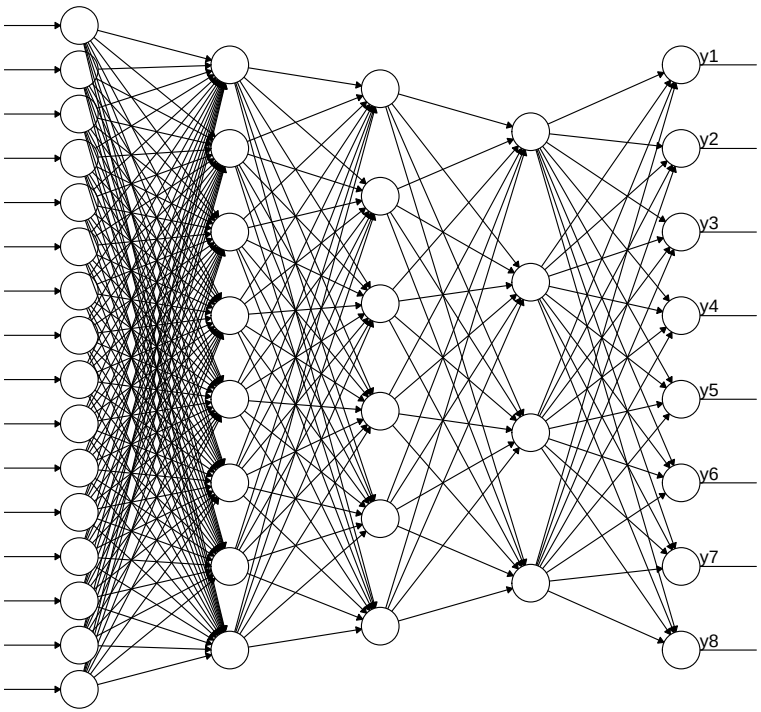
   4. Change the weights w_{ij} to minimise the error's cost $c(x, z(y))$.

Feed-forward neural networks



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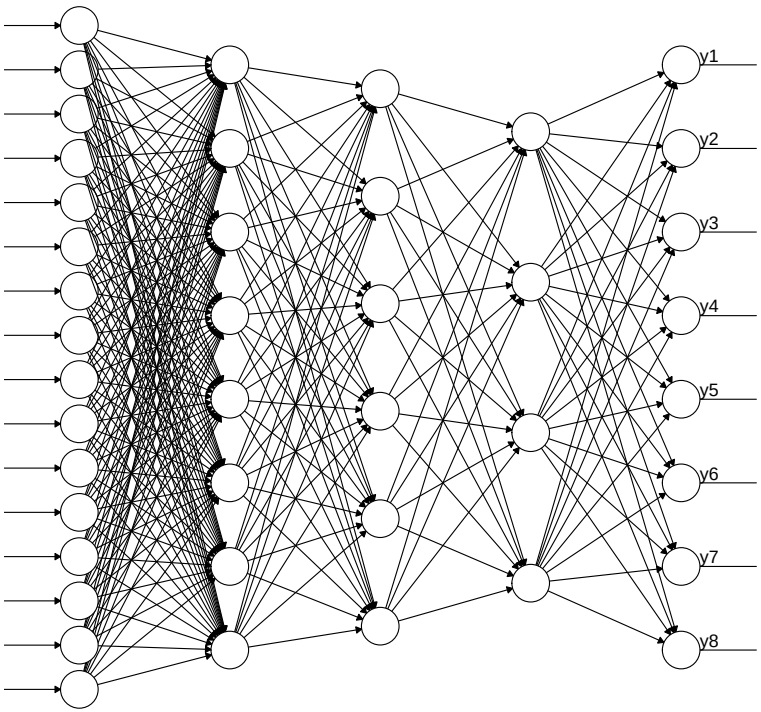

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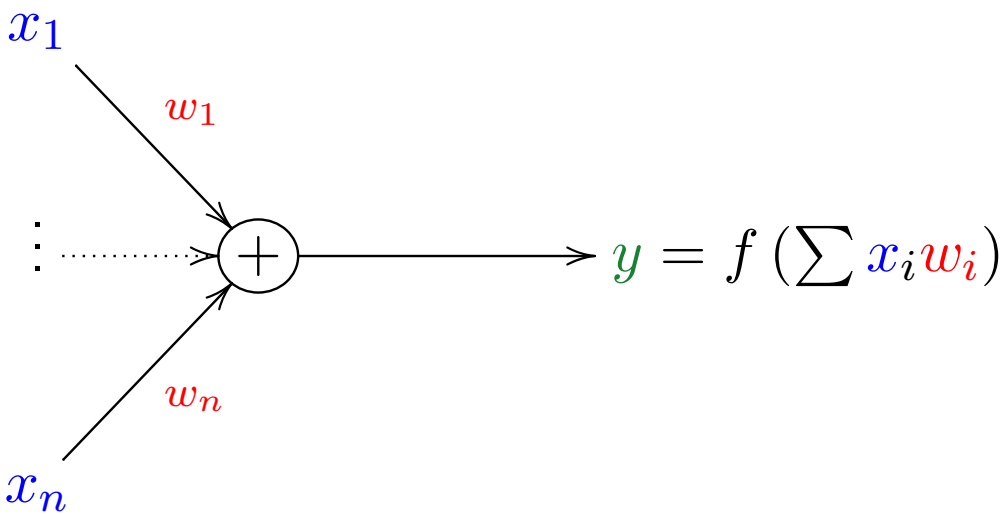
Integrate and Fire Neurons

McCulloch and Pitts (1943)



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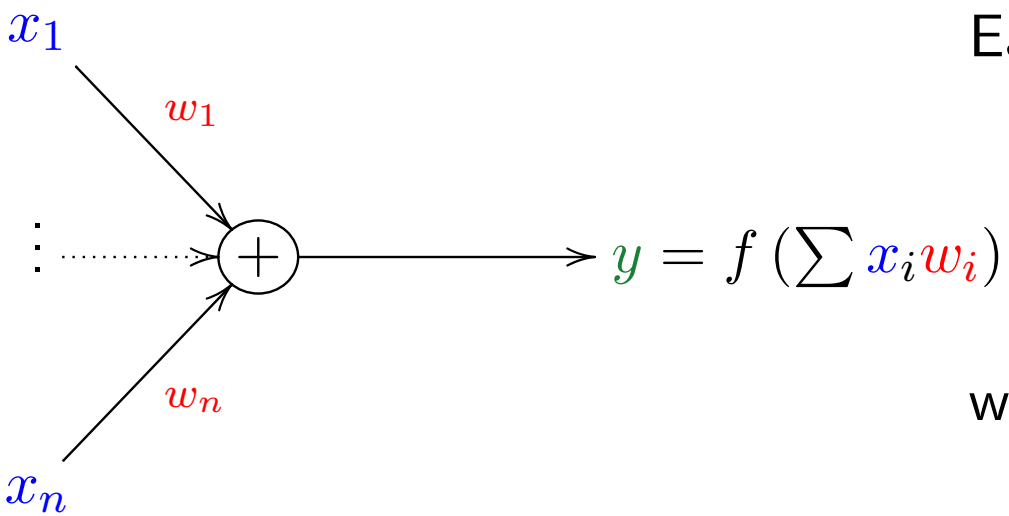

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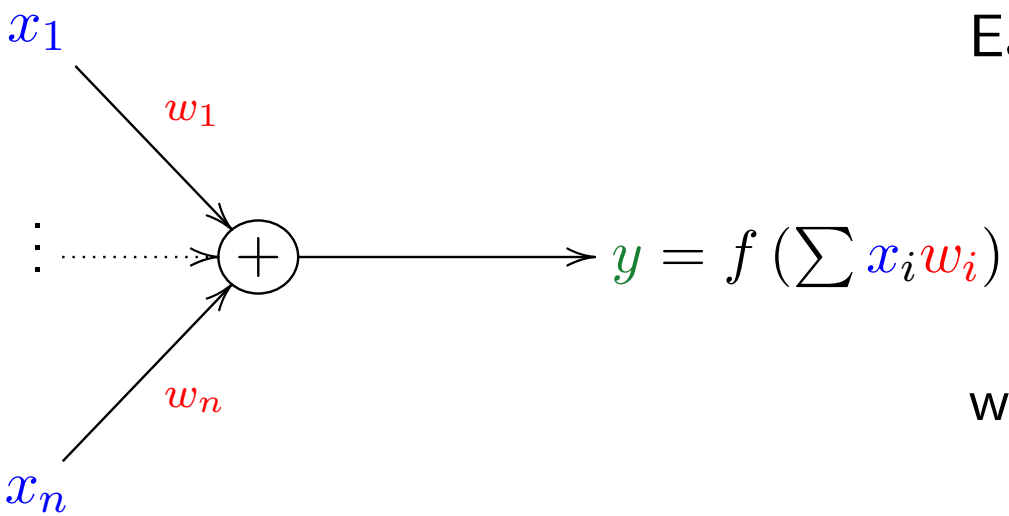

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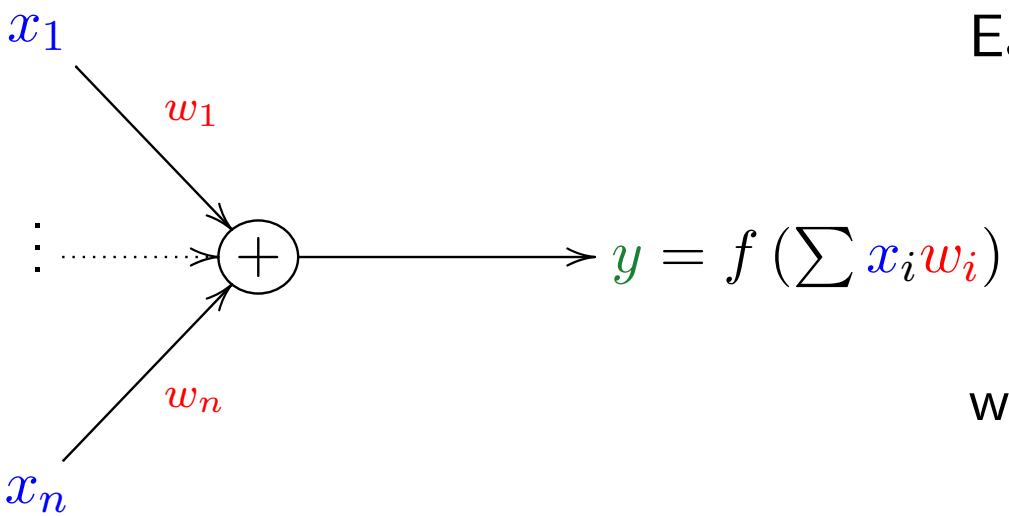

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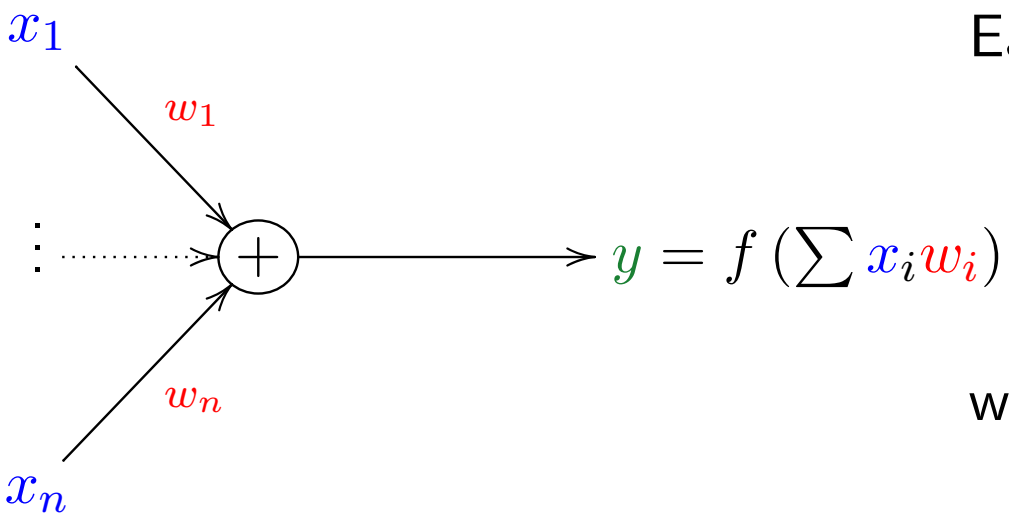

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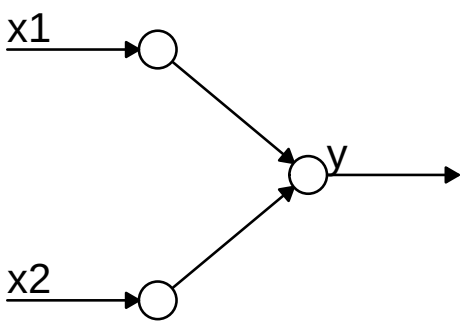

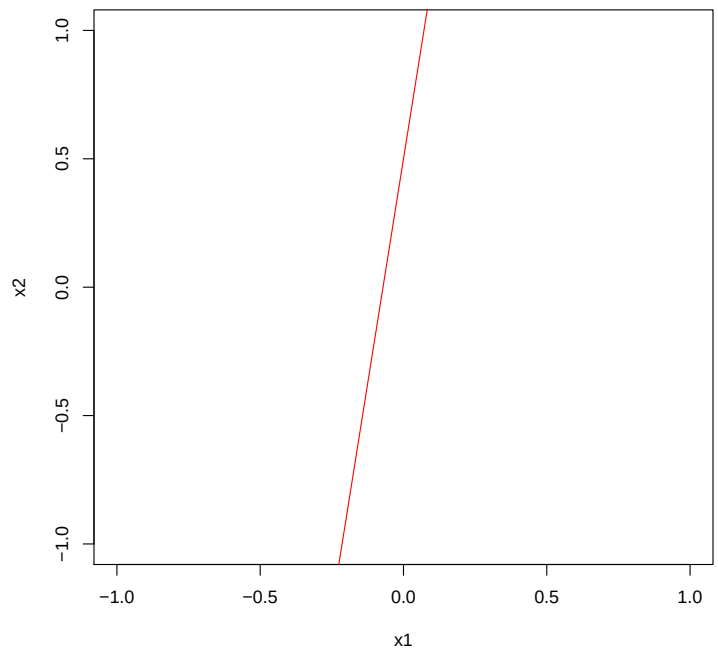

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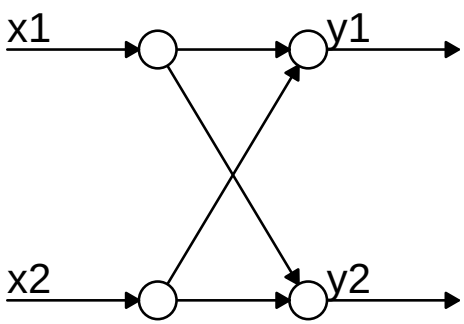

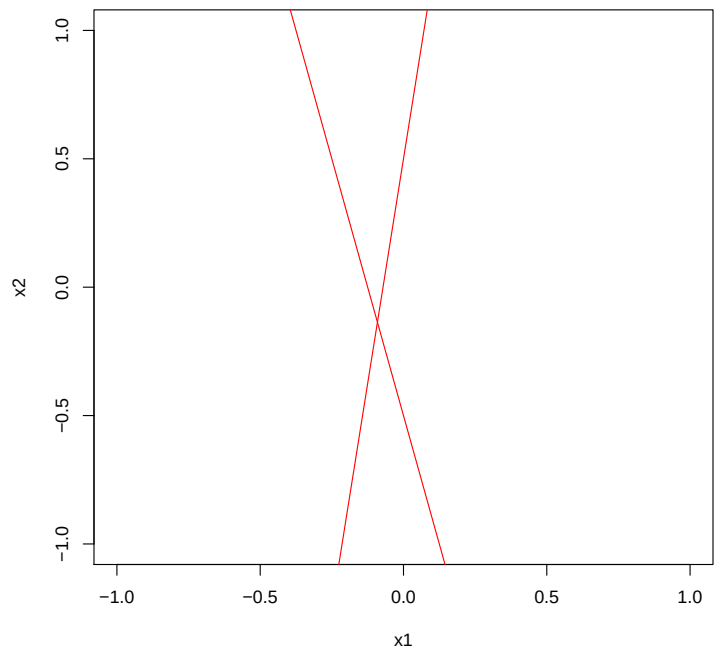

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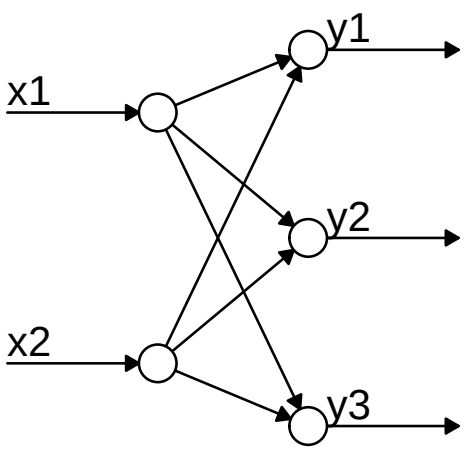

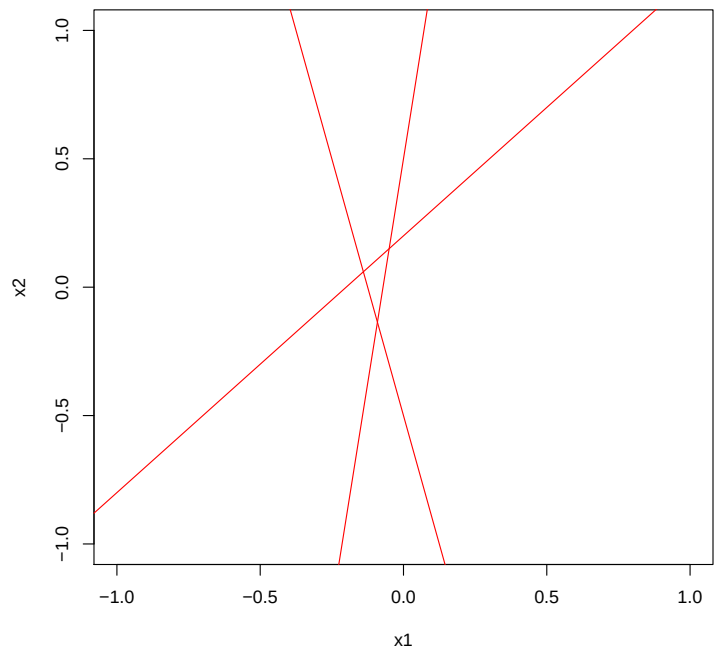

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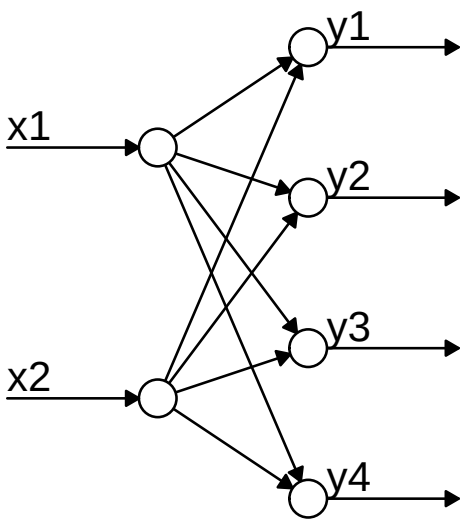

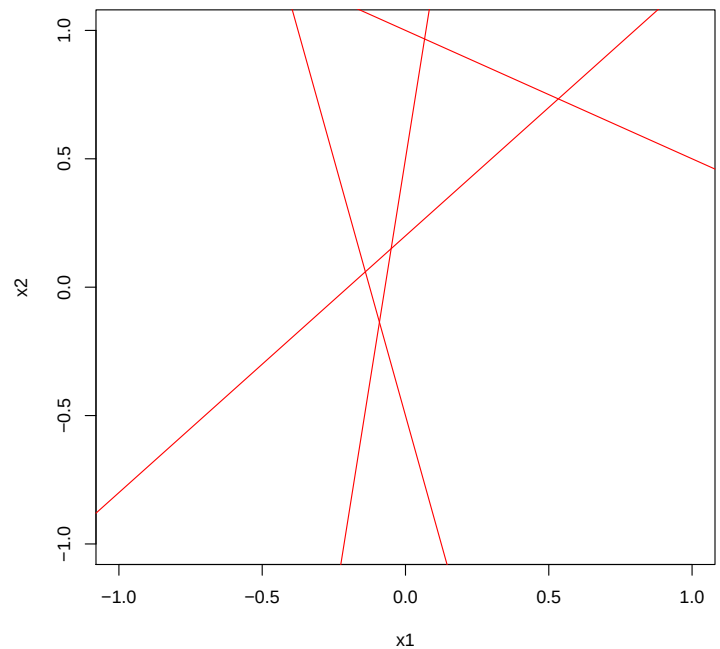

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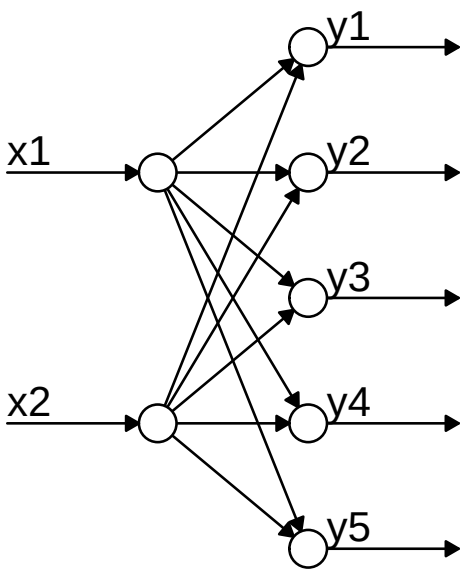

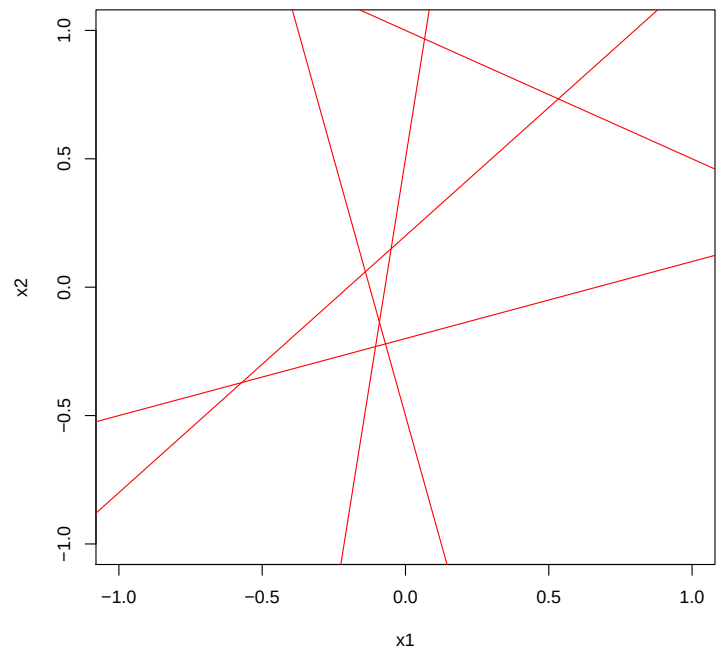

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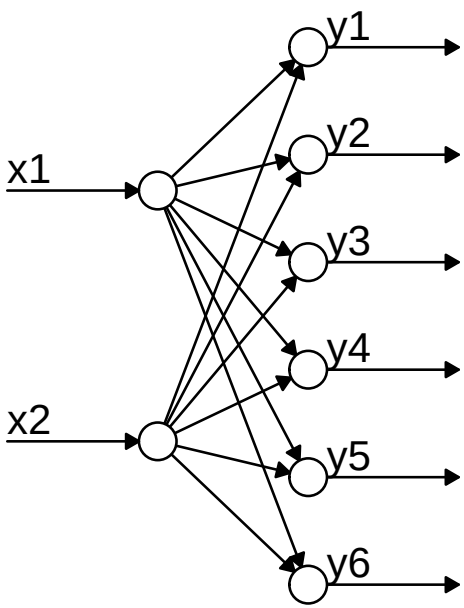

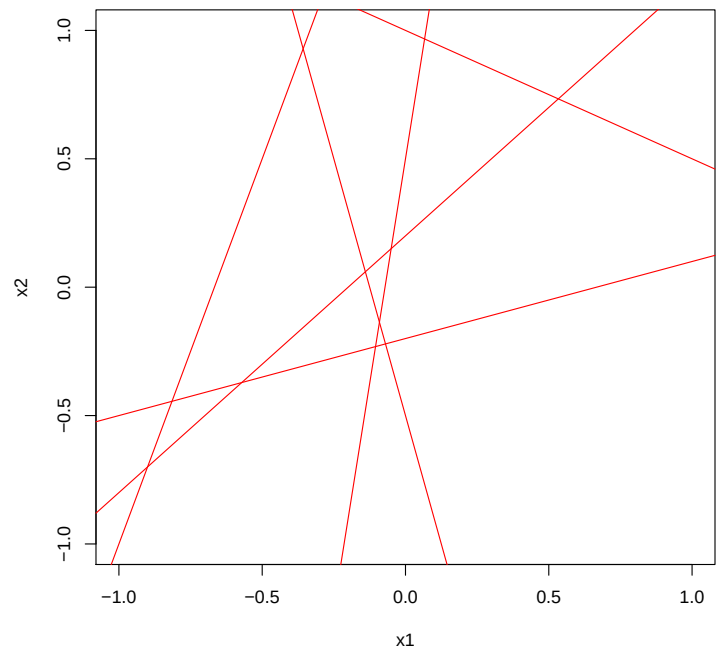

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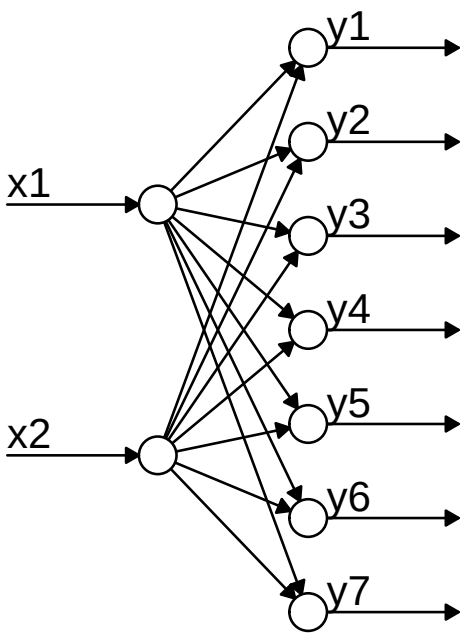

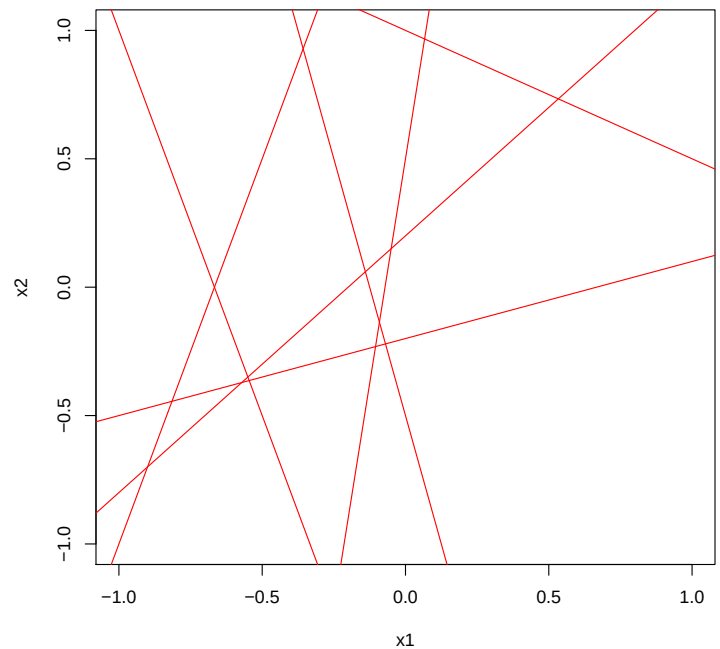

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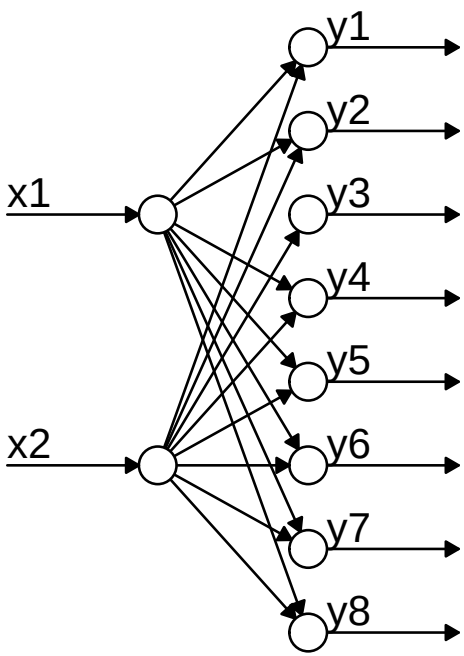

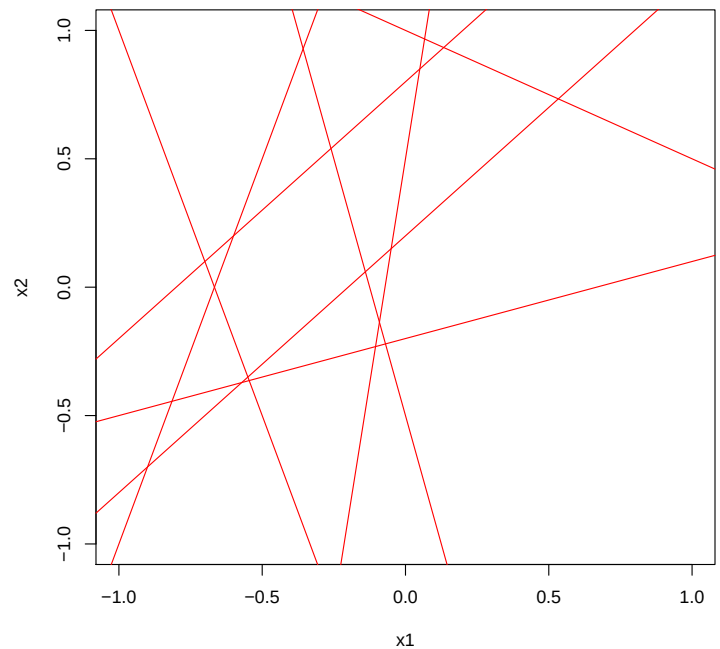

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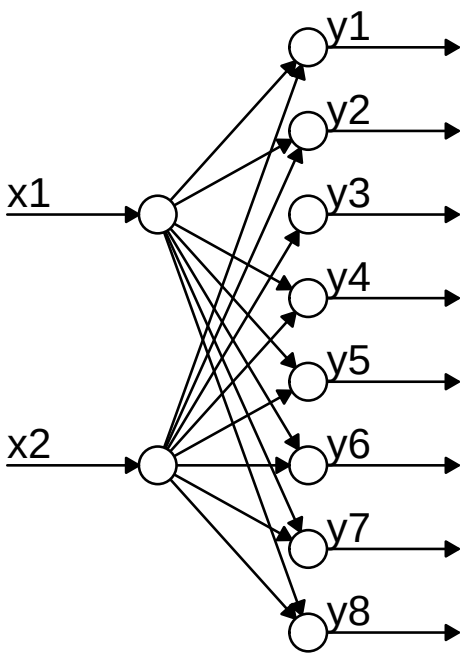

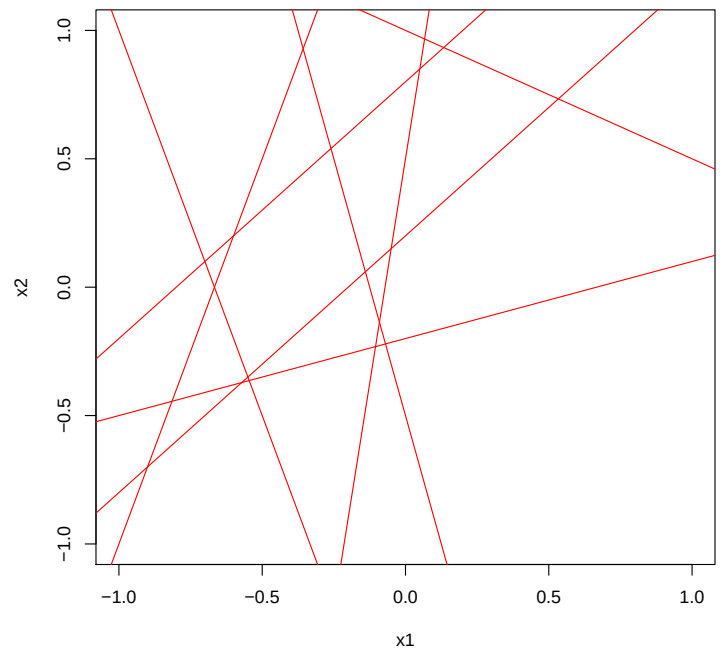

- k nodes partition the space into 2^k subsets.

Learning neural models and VoI

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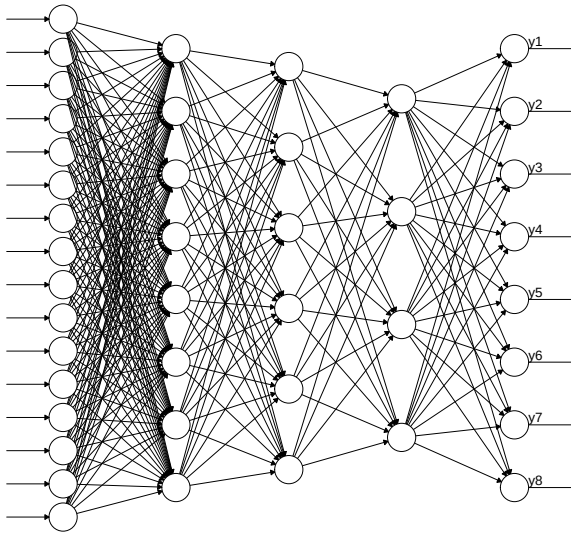

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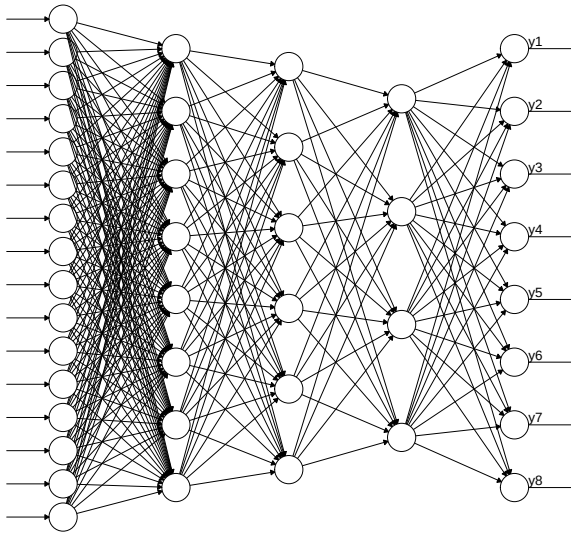

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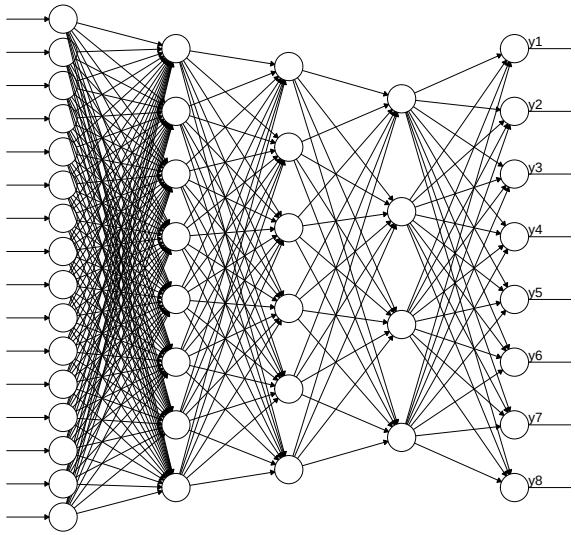

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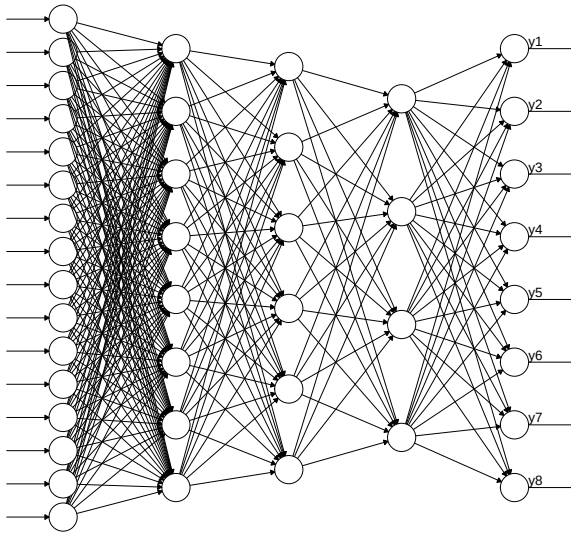

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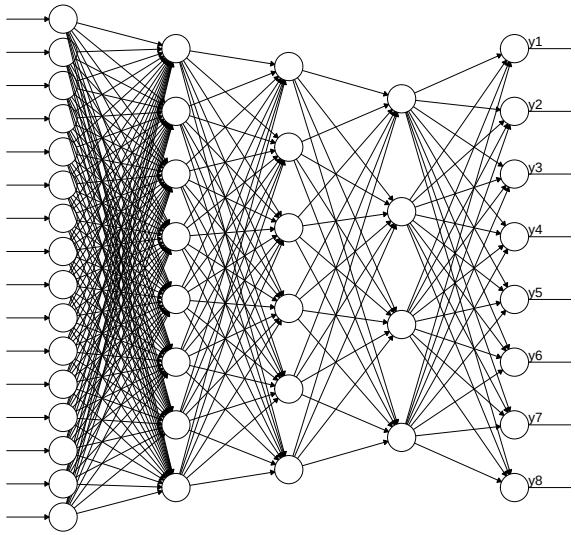

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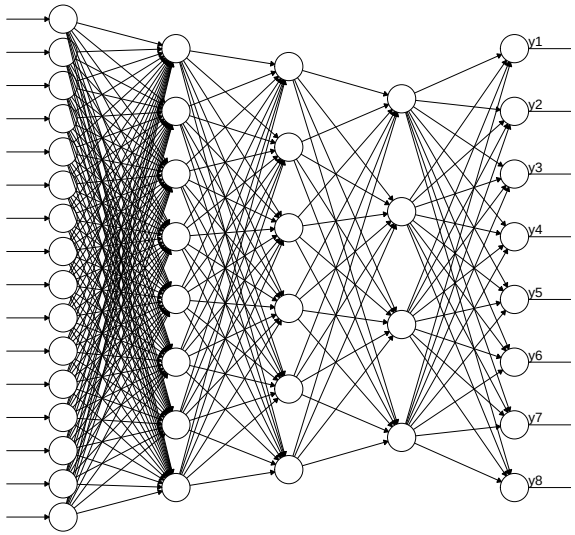

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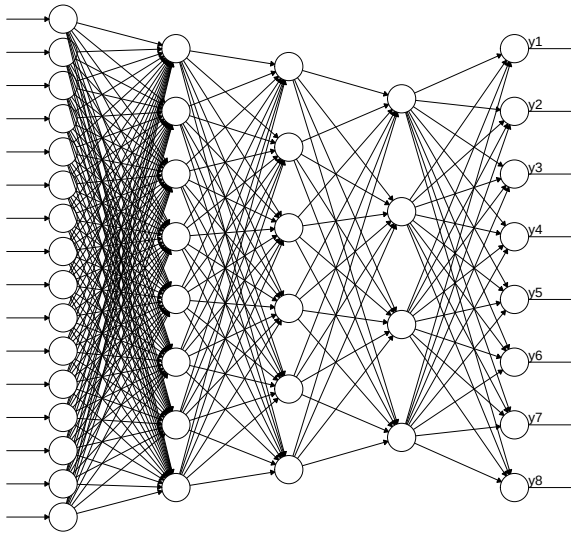

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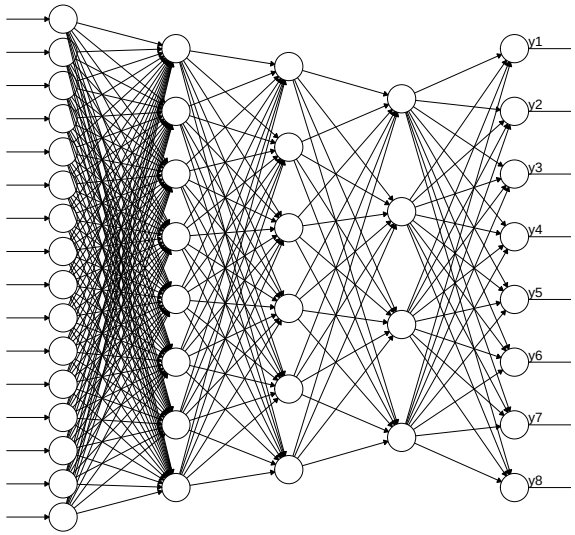

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Linear trend and forecast

Table: TSLA.Close price Aug 2020

Date	Price(t)	
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2020-08-05	297.004	
2020-08-06	297.916	
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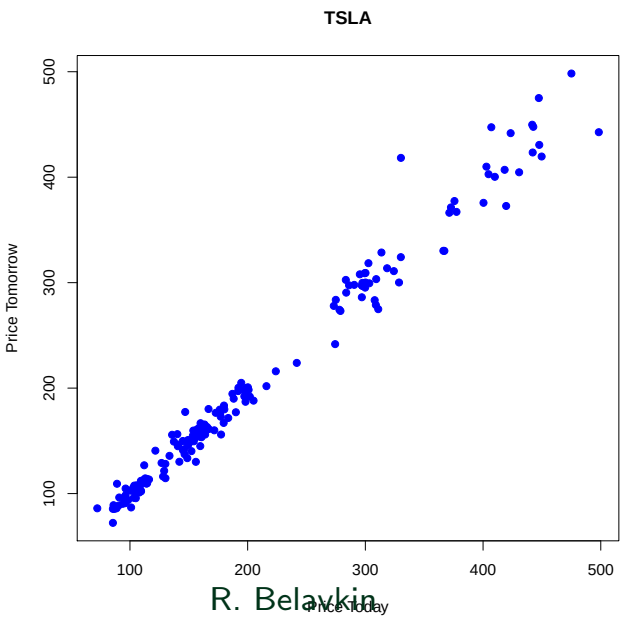

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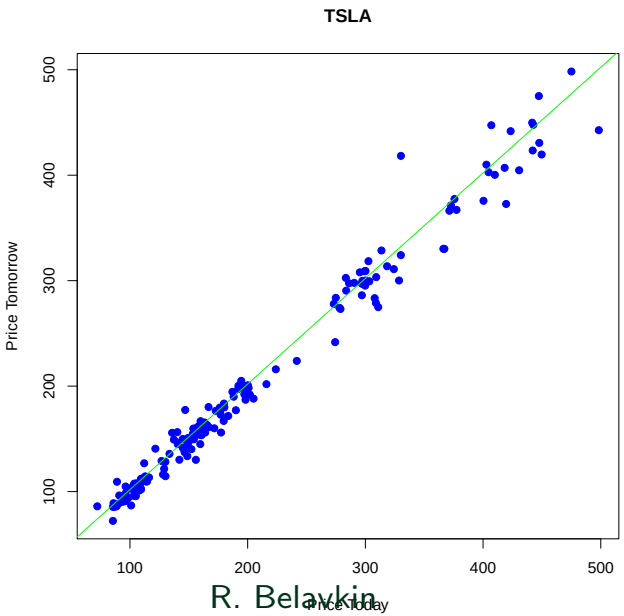

Linear trend and forecast

Table: TSLA.Close price Aug 2020

Date	Price(t)	Price(t+1)
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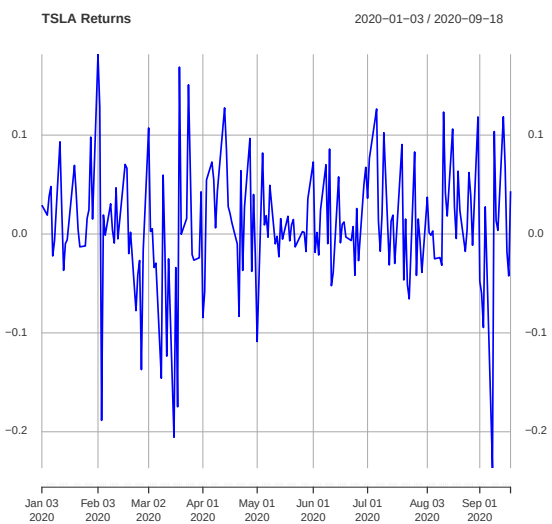

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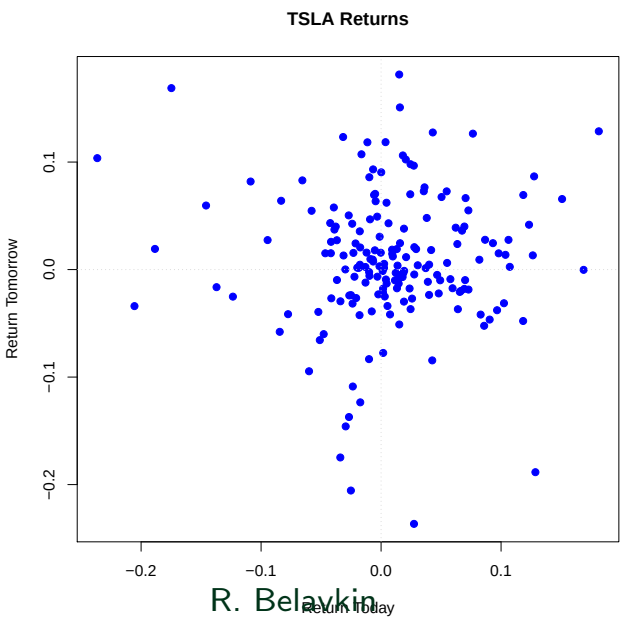

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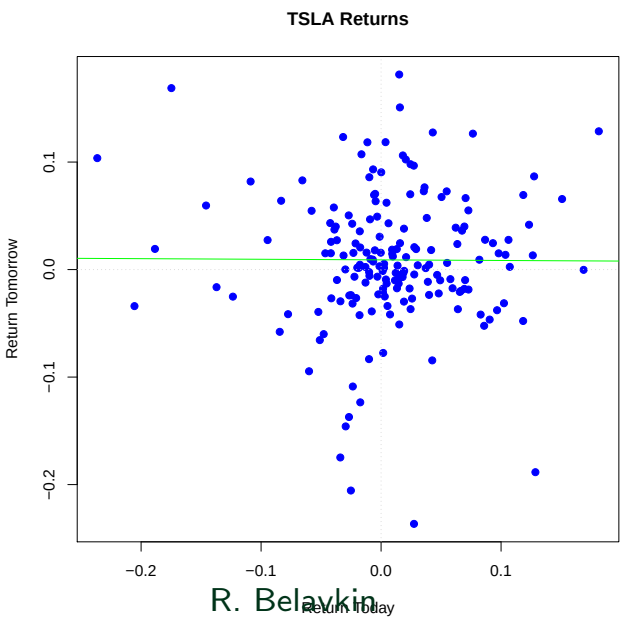

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Table: BTCUSD log-returns Jan 2019

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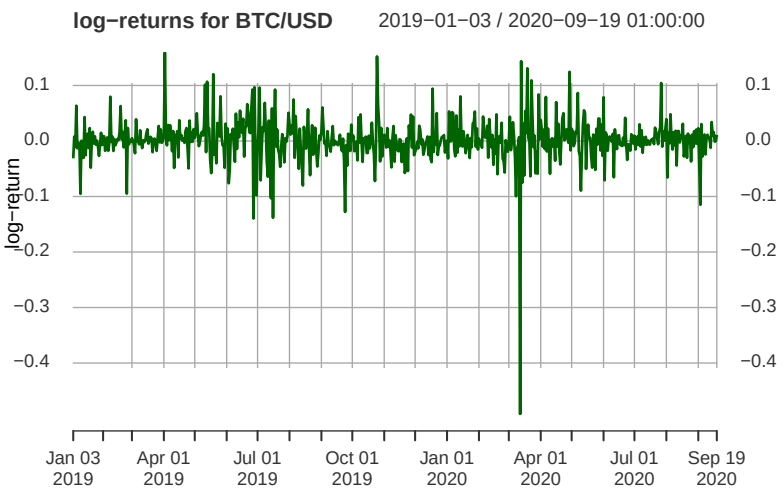

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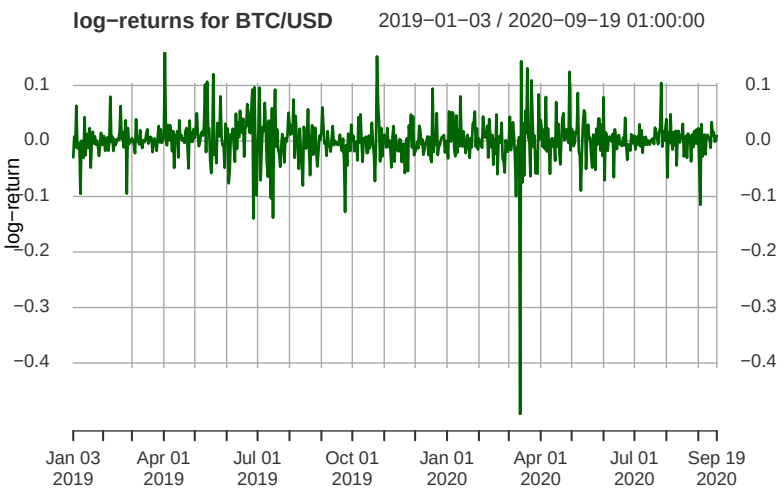

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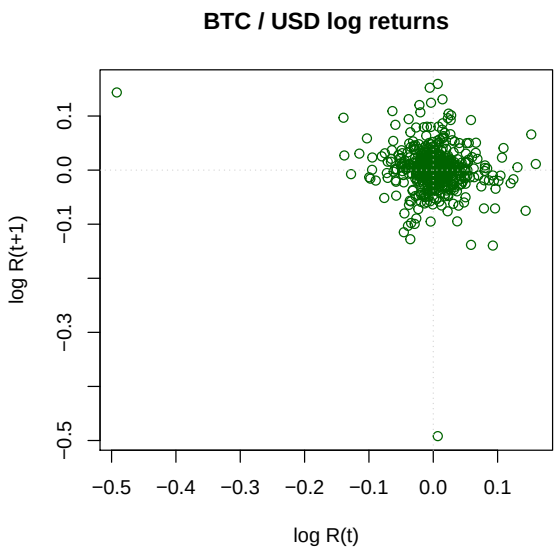

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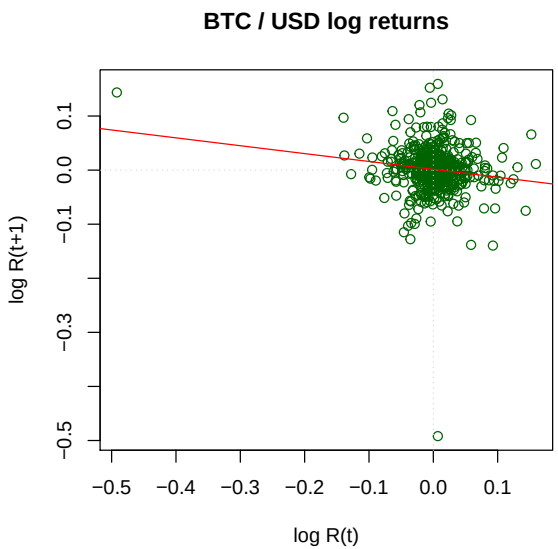

- We can try to model log-returns for two consecutive days:

$$\log R(t+1) \approx a + b \log R(t)$$

where $R(t) = S(t)/S(t-1)$ ratio of prices.

- The performance may be poor, as the correlation is small.

Predicting log-returns (cont.)

Table: BTCUSD log-returns Jan 2019

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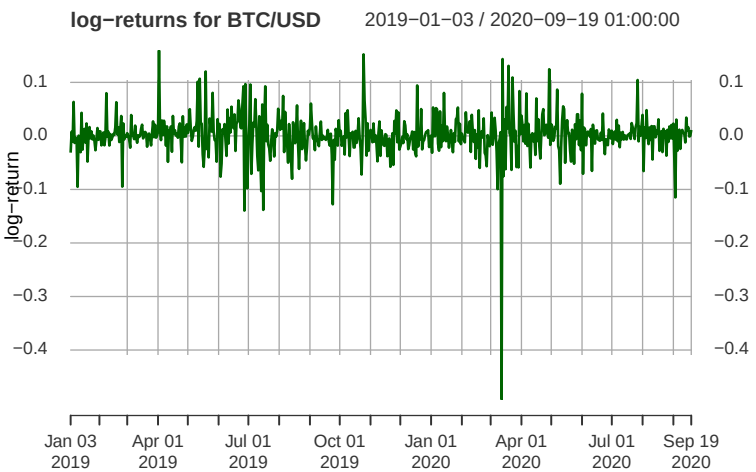

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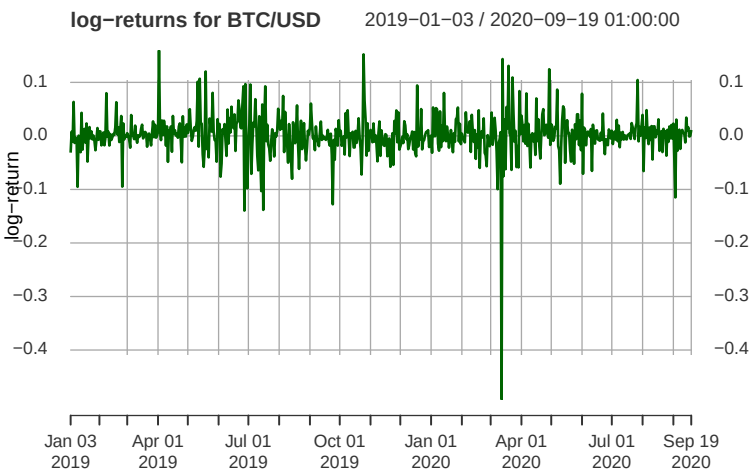

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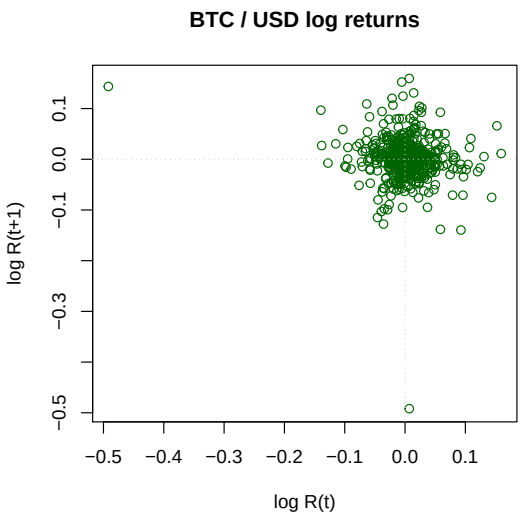
Predict next day log-return based on n previous day's log-returns:

$$\log R(t+1) \approx a + b_1 \log R(t) + b_2 \log R(t-1) + \cdots + b_n \log R(t-n)$$

where $R(t) = S(t)/S(t-1)$ ratio of prices.



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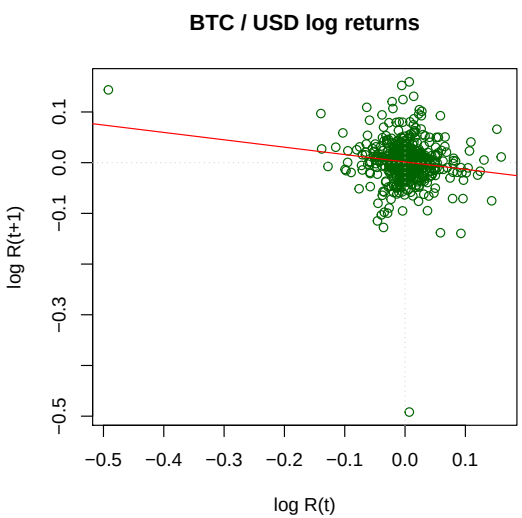

We may also include past log-returns of other cryptocurrencies as predictors.

Application to trading

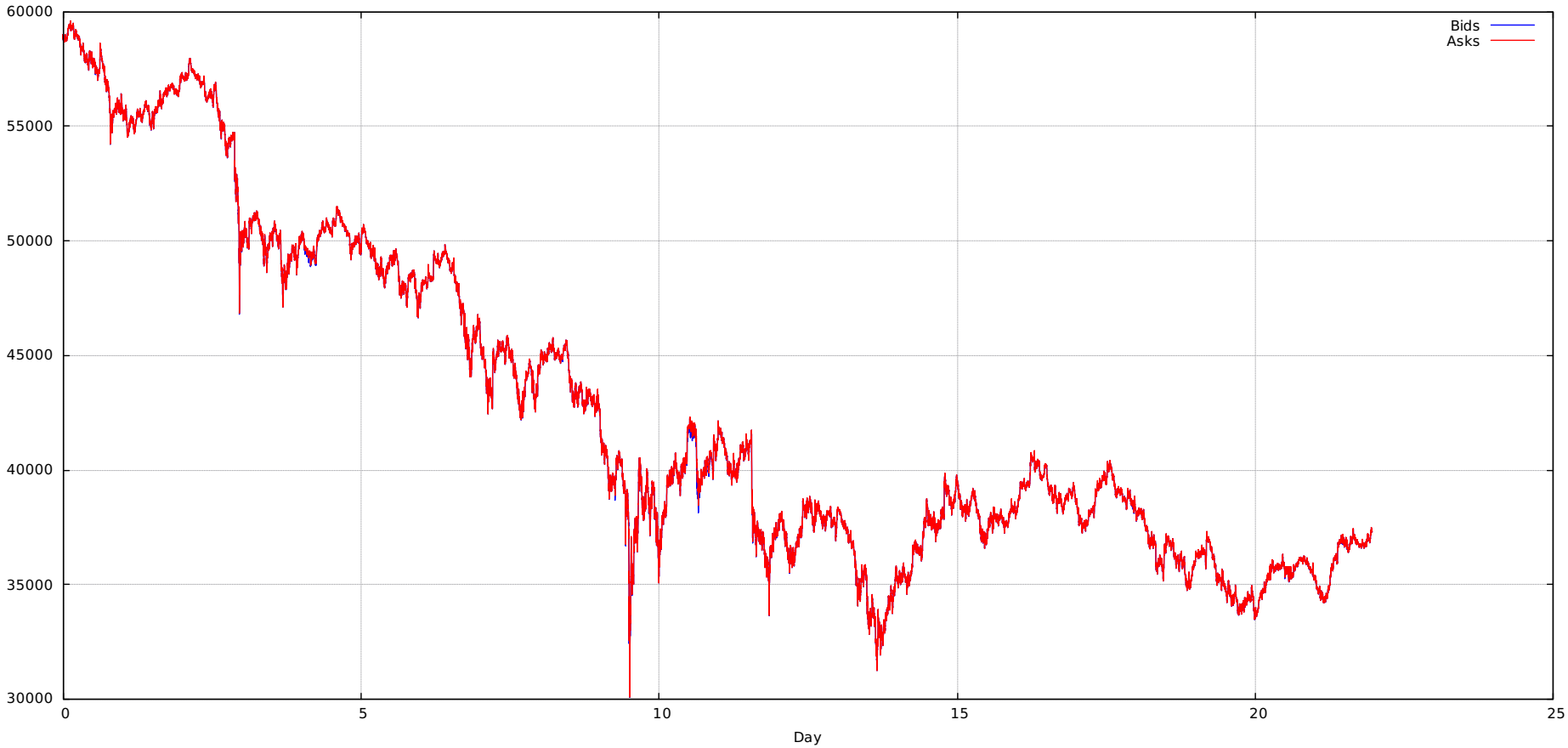

Figure: BTC/USD, 10–22 May, 2021

Application to trading

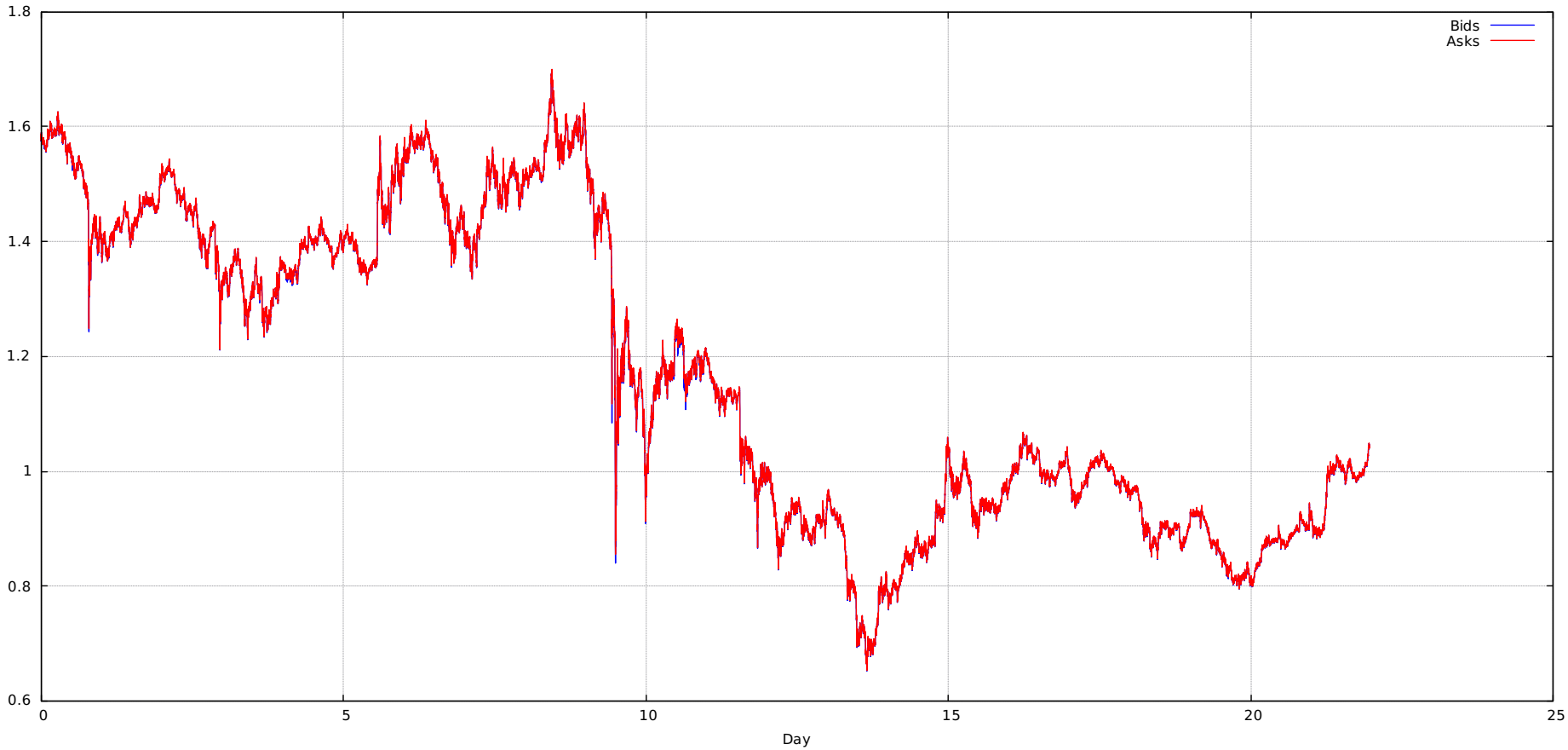

Figure: XRP/USD, 10–22 May, 2021

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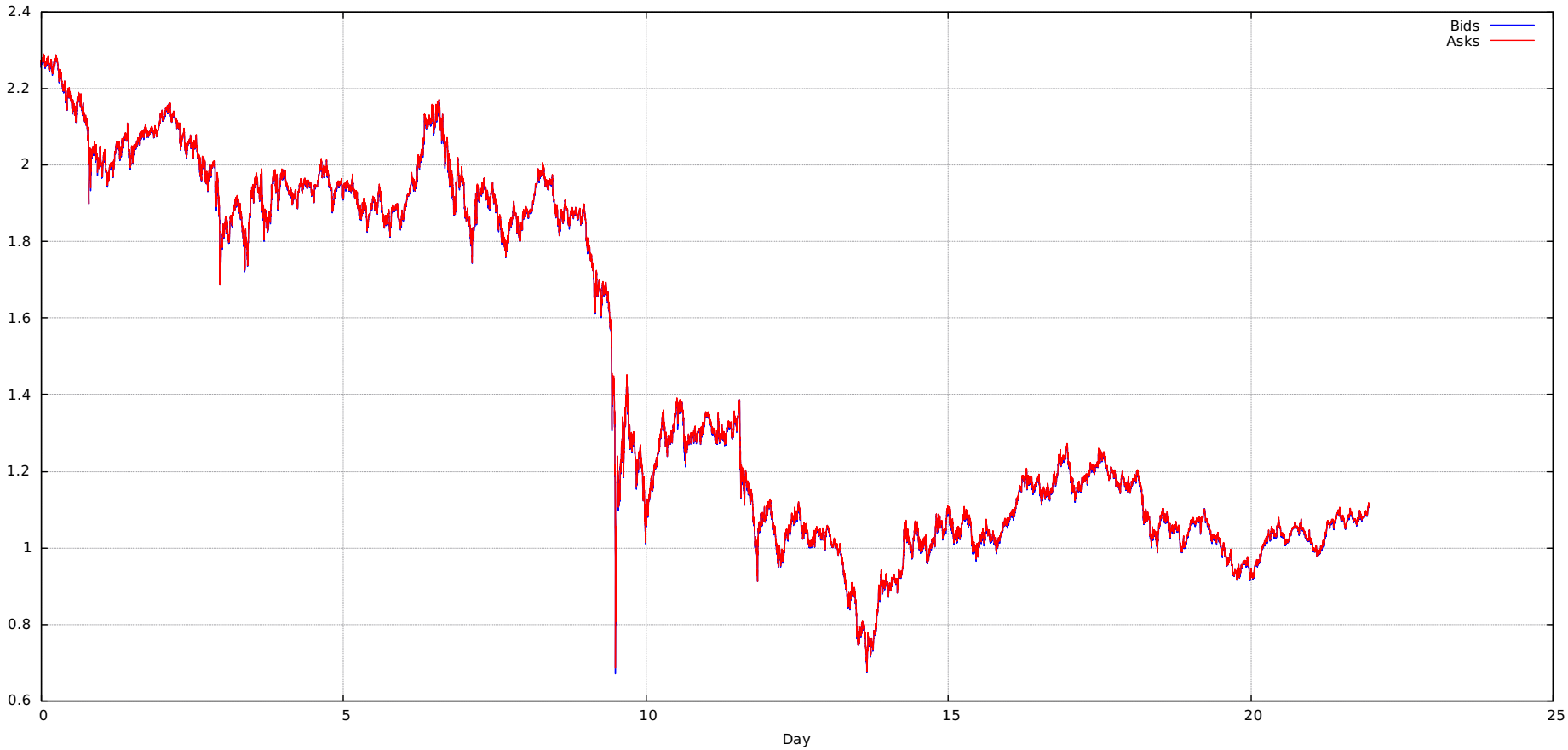

Figure: IOT/USD, 10–22 May, 2021

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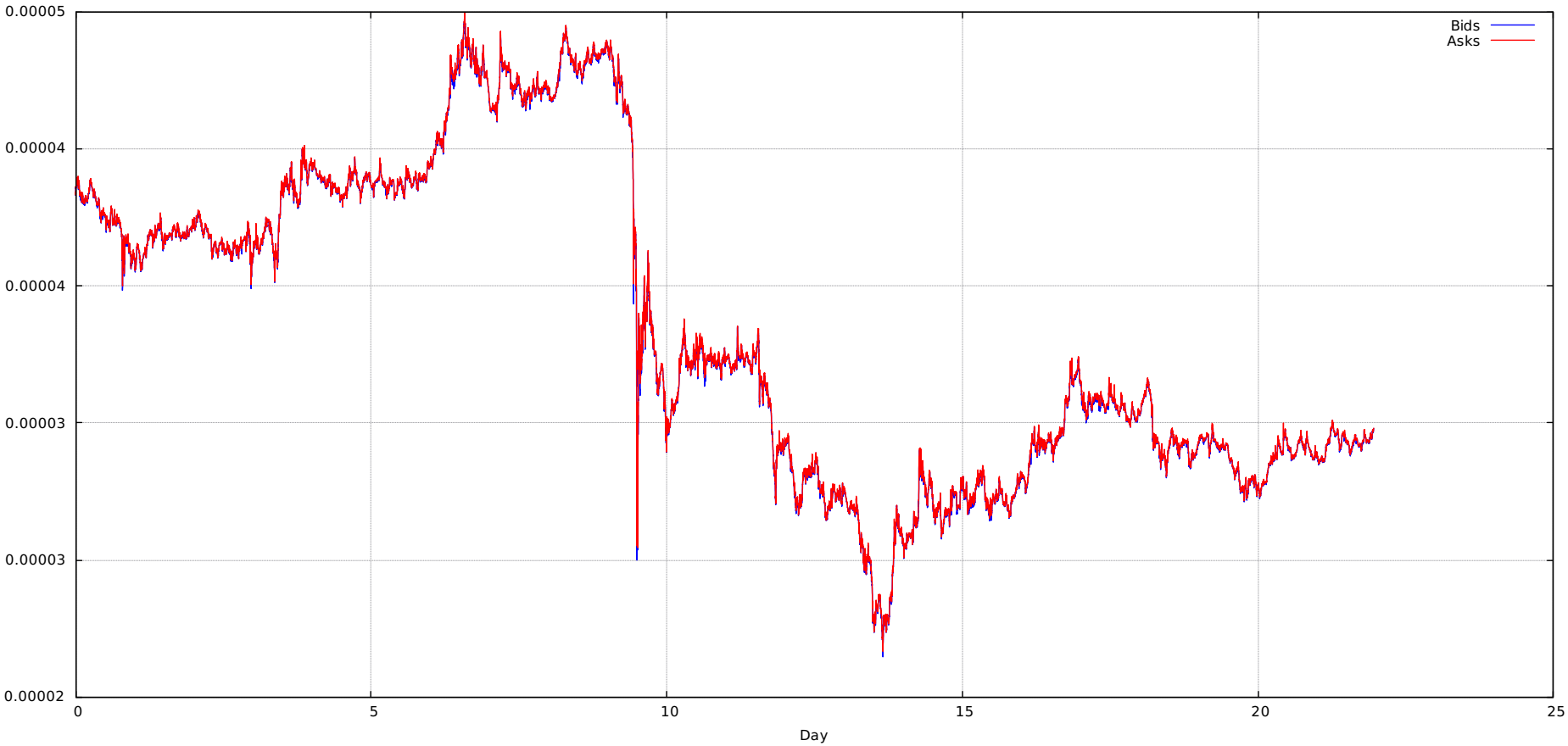

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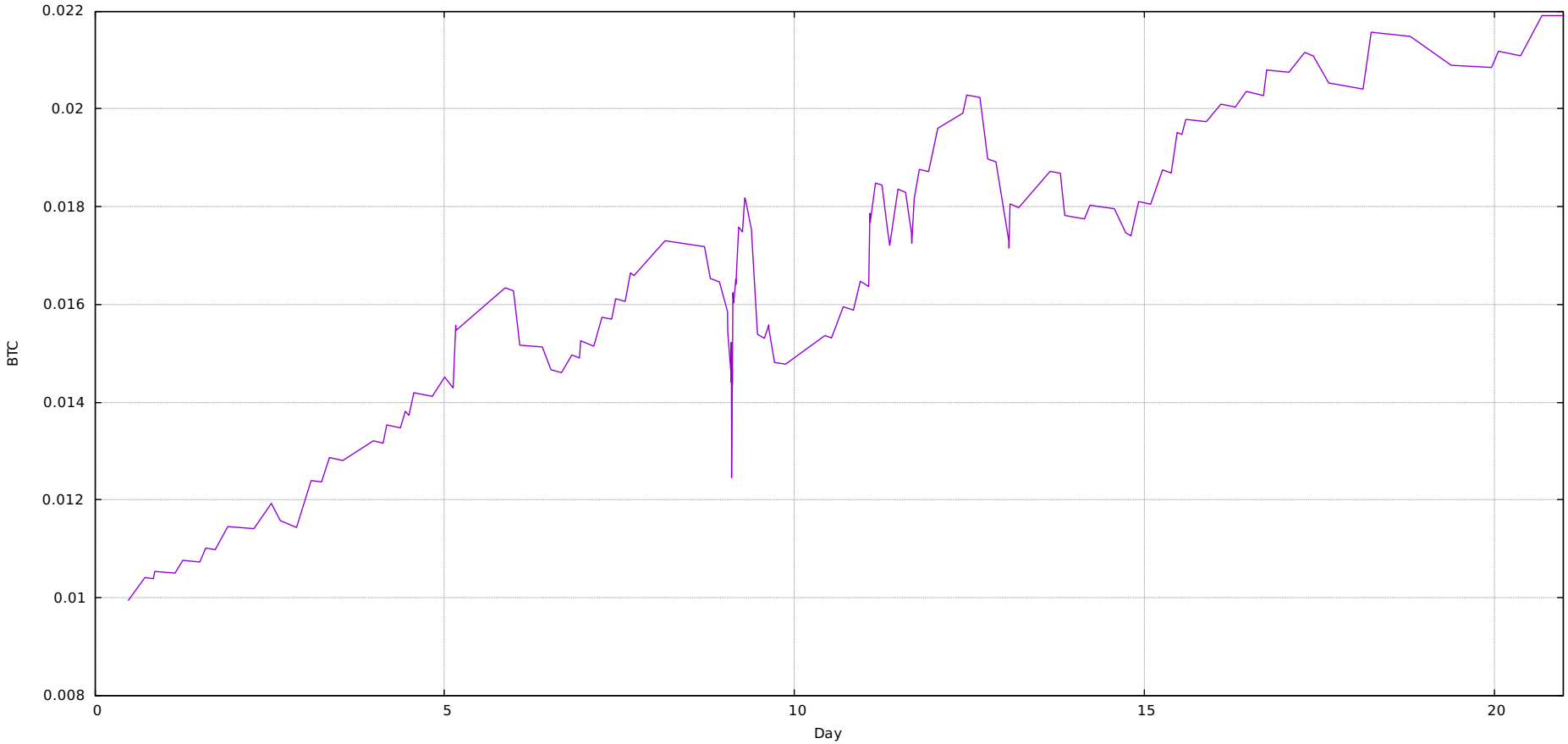

Figure: Amount of BTC, trading during 10–22 May, 2021, using array of forecasts for ≈ 300 trade pairs and ranging from 30 min to 24h into the future.

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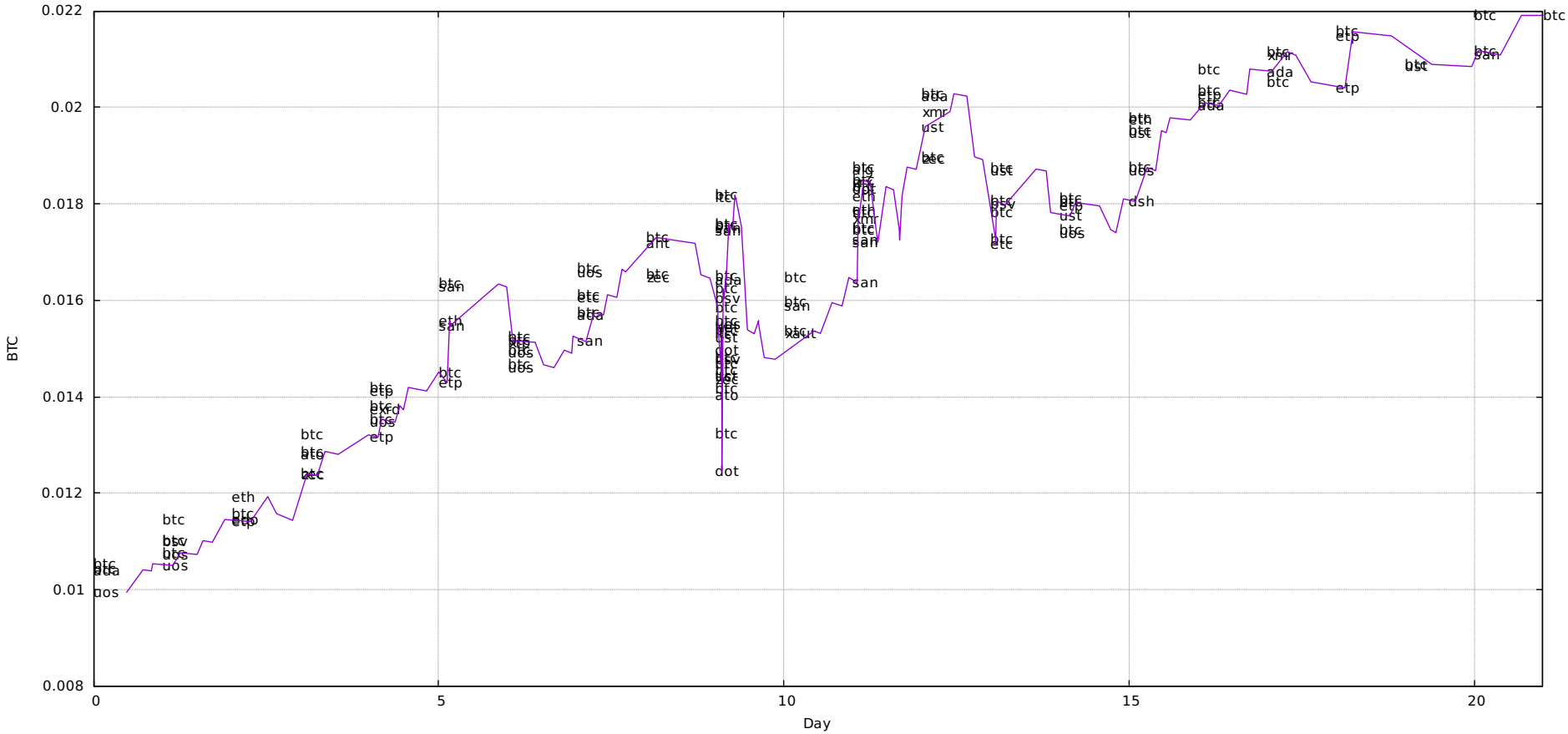

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- *Entropy* as the expected 'surprise' $-\ln P(z)$:

$$H(Z) := \mathbb{E}_{P(z)}\{-\ln P(z)\}$$

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Remark

We are looking for optimal function $z(x) = z \circ y(x)$ *subject to* $H(Z) \leq \ln k$.

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- Shannon's *mutual information*:

$$I(X,Z) := \mathbb{E}_{P(x,z)} \left\{ \ln \frac{P(z \mid x)}{P(z)} \right\}$$

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where $P(z \mid x) = \sum_Y P(z \mid y)P(y \mid x)$ and $I(X,Y) \leq I$.

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Remark

Instead of functions $z(x) = z \circ y(x)$, we are looking for optimal $P(z \mid x)$ subject to $I(X,Z) \leq I$.

Asymptotic Equivalence of Different VoIs

- x — hidden, y — observed, z — control, $u(x, z)$ — utility.

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$$P(x) \qquad \mathbb{E}\{u(x, z)\}$$

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$$P(x) \qquad \max_z \mathbb{E}\{u(x, z)\} =: U(0)$$

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$$P(x \mid y) \qquad \max_{P(z|x):I\{x,z\}\leq\lambda} \mathbb{E}\{u(x, z) \mid y\}$$

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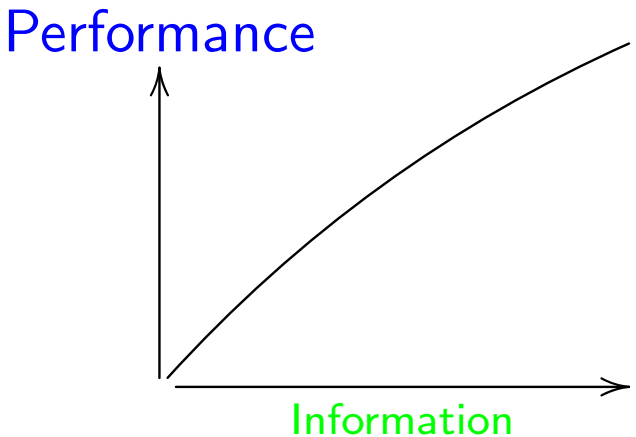
- One of the main results of the theory is that all types of VoI are asymptotically equivalent (Stratonovich, 1975).

The Maximum Entropy (or first) variational problem

Problem I

- Linear programming problem:

$$\text{maximize (minimize)} \quad \mathbb{E}_p\{u\} \quad \text{subject to} \quad \mathbb{E}_p\{\ln(p/q)\} \leq \lambda$$



Maximize performance

s.t. information $\leq \lambda$

The Maximum Entropy (or first) variational problem

Problem I

- Linear programming problem:

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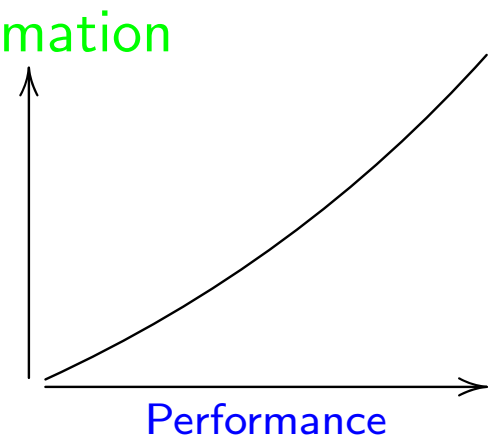
Minimize information

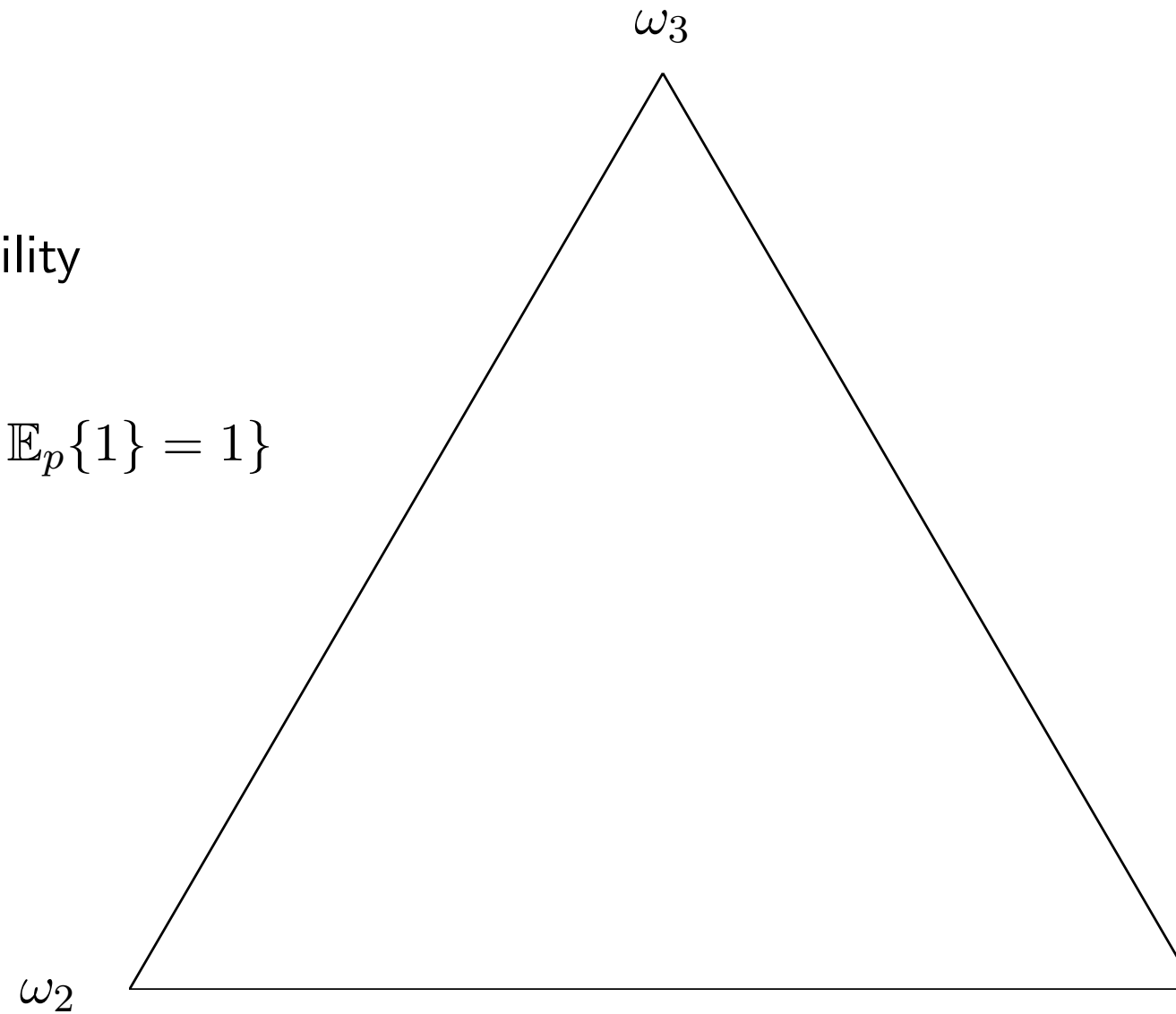
s.t. performance $\ge \upsilon$



Information-geometruc view

- The set of all probability measures

$$\mathcal{P}(\Omega) := \{p : p \geq 0\,,\ \mathbb{E}_p\{1\} = 1\}$$



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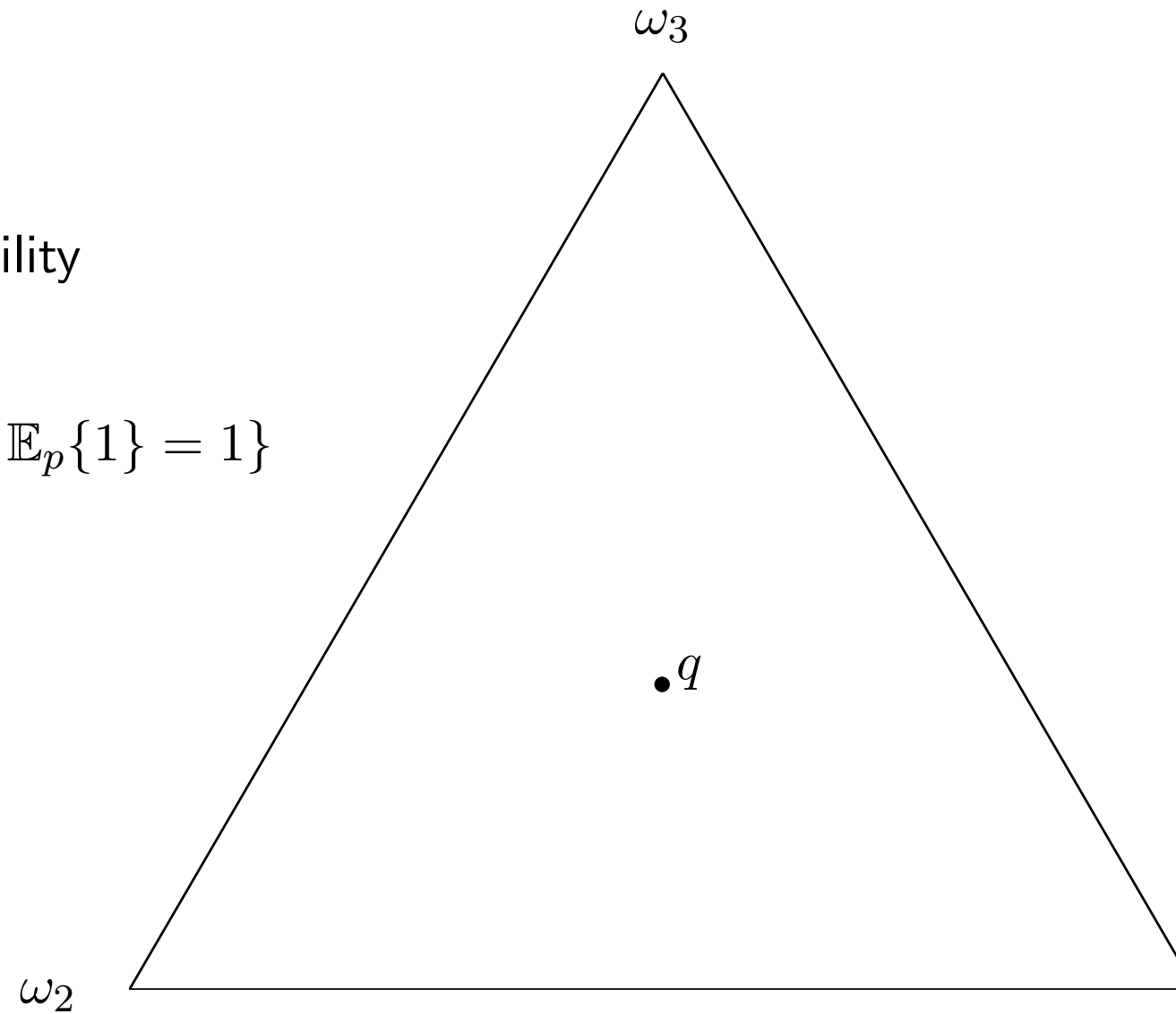

Information-geometruc view

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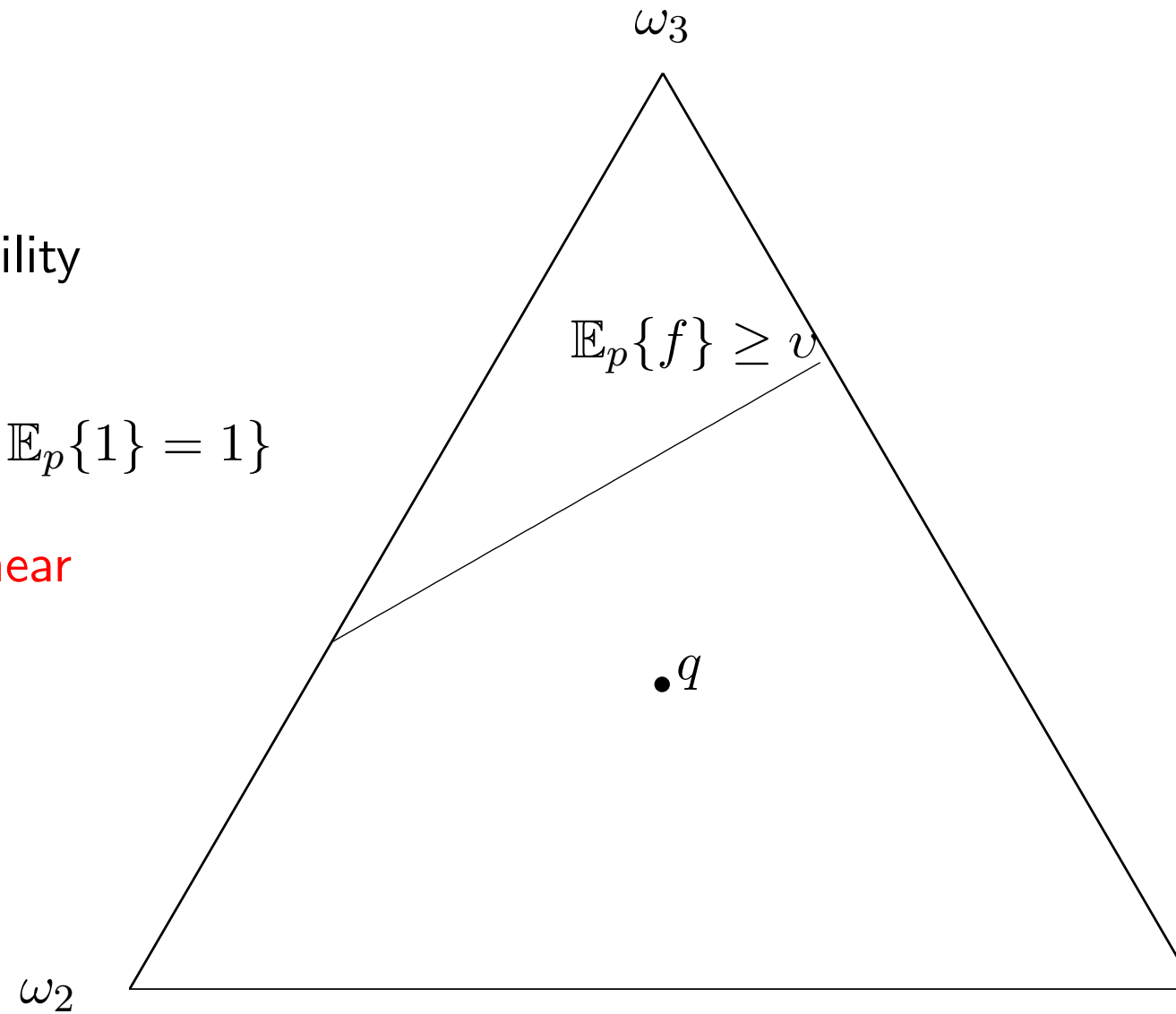
  $\mathcal{P}(\Omega) := \{p : p \geq 0\,,\ \mathbb{E}_p\{1\} = 1\}$

- $\mathbb{E}_p\{u\} := \langle u, p\rangle$ is linear



Information-geometruc view

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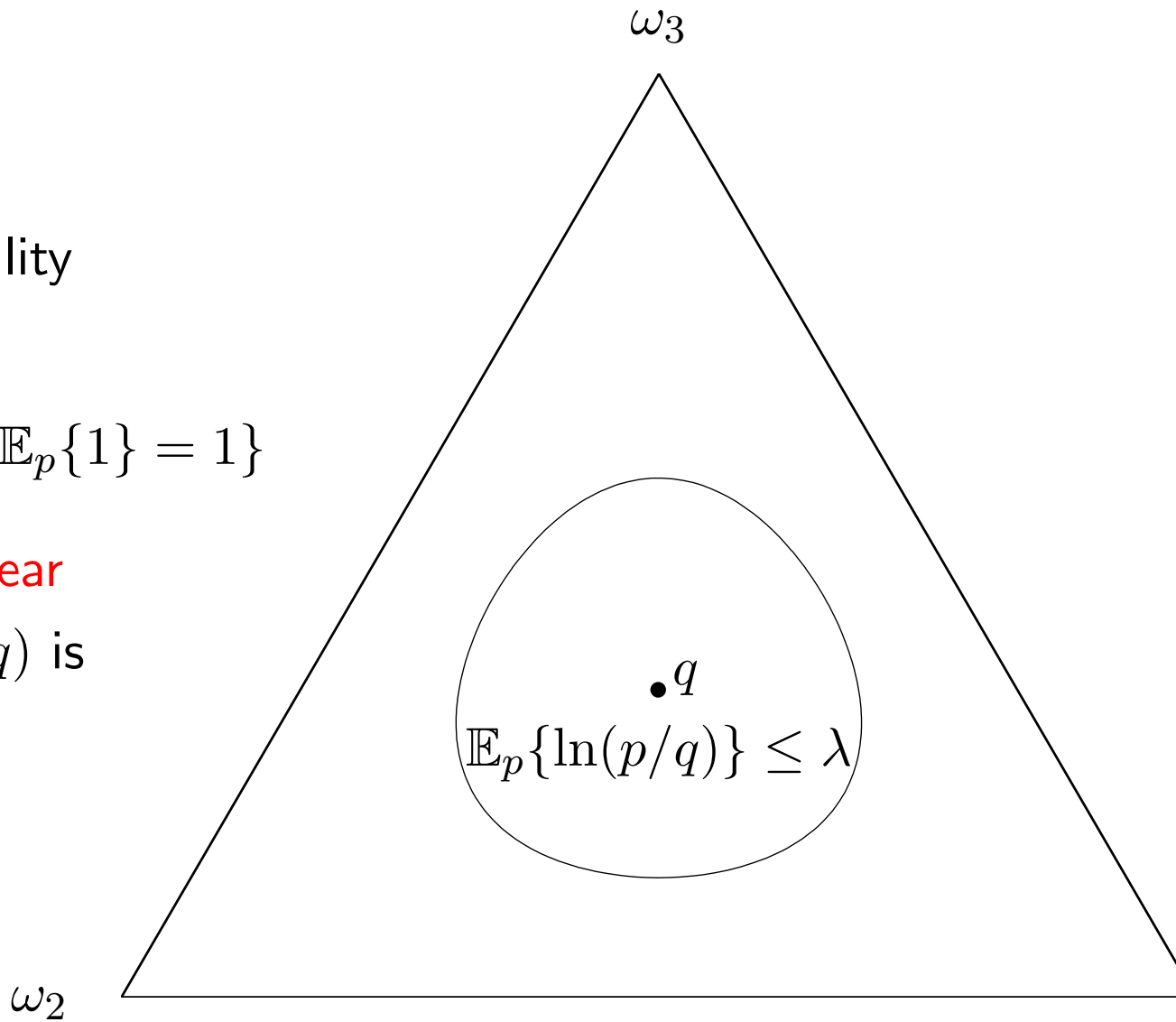
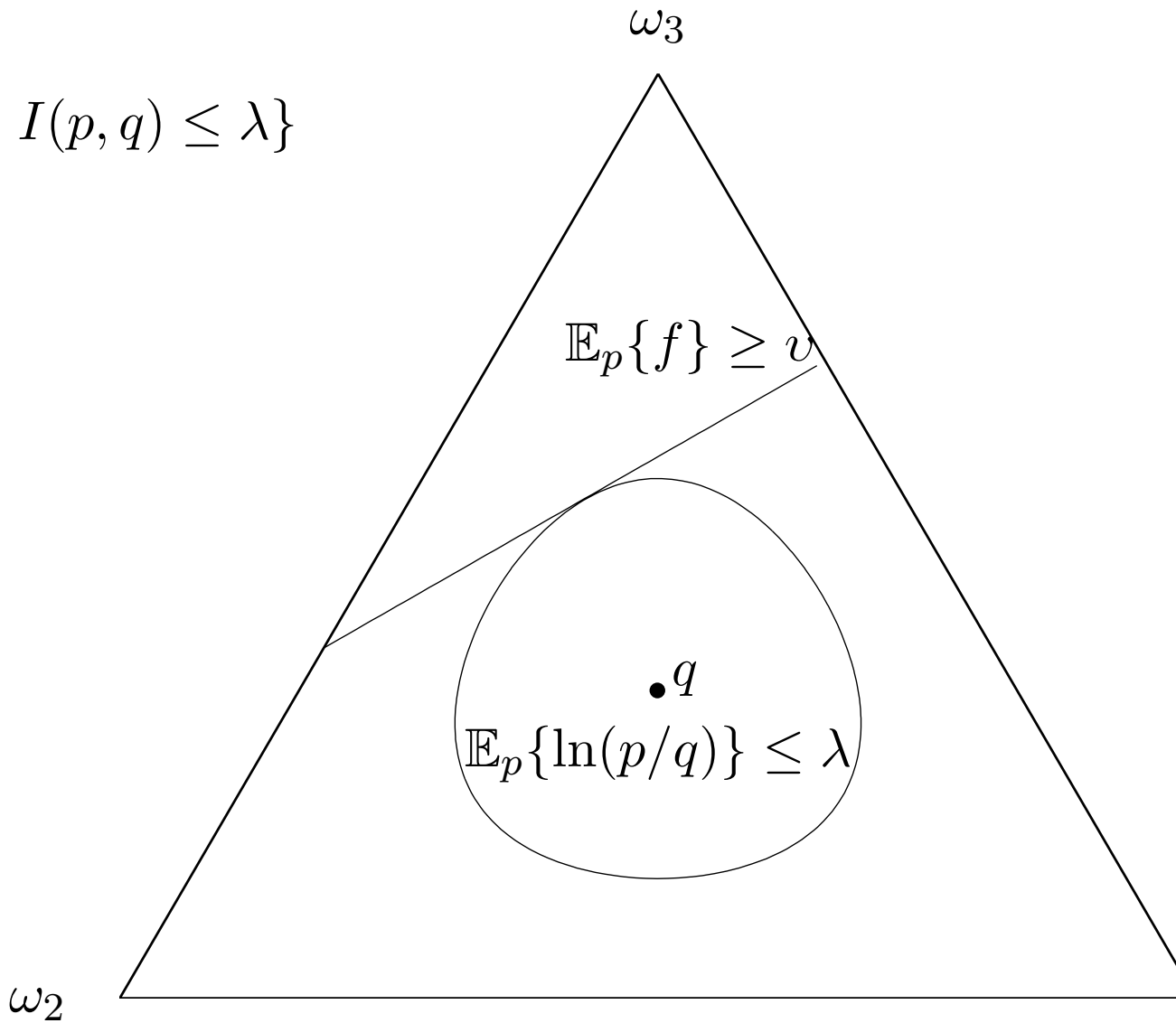
- $\mathbb{E}_p\{\ln(p/q)\} =: I(p,q)$ is convex



Information-geometruc view (cont)

- Maximize $\mathbb{E}_p\{u\}$

$$V(\lambda) := \sup\{\mathbb{E}_p\{u\} : I(p,q) \leq \lambda\}$$



Information-geometruc view (cont)

- Maximize $\mathbb{E}_p\{u\}$

  $V(\lambda) := \sup\{\mathbb{E}_p\{u\} : I(p,q) \leq \lambda\}$

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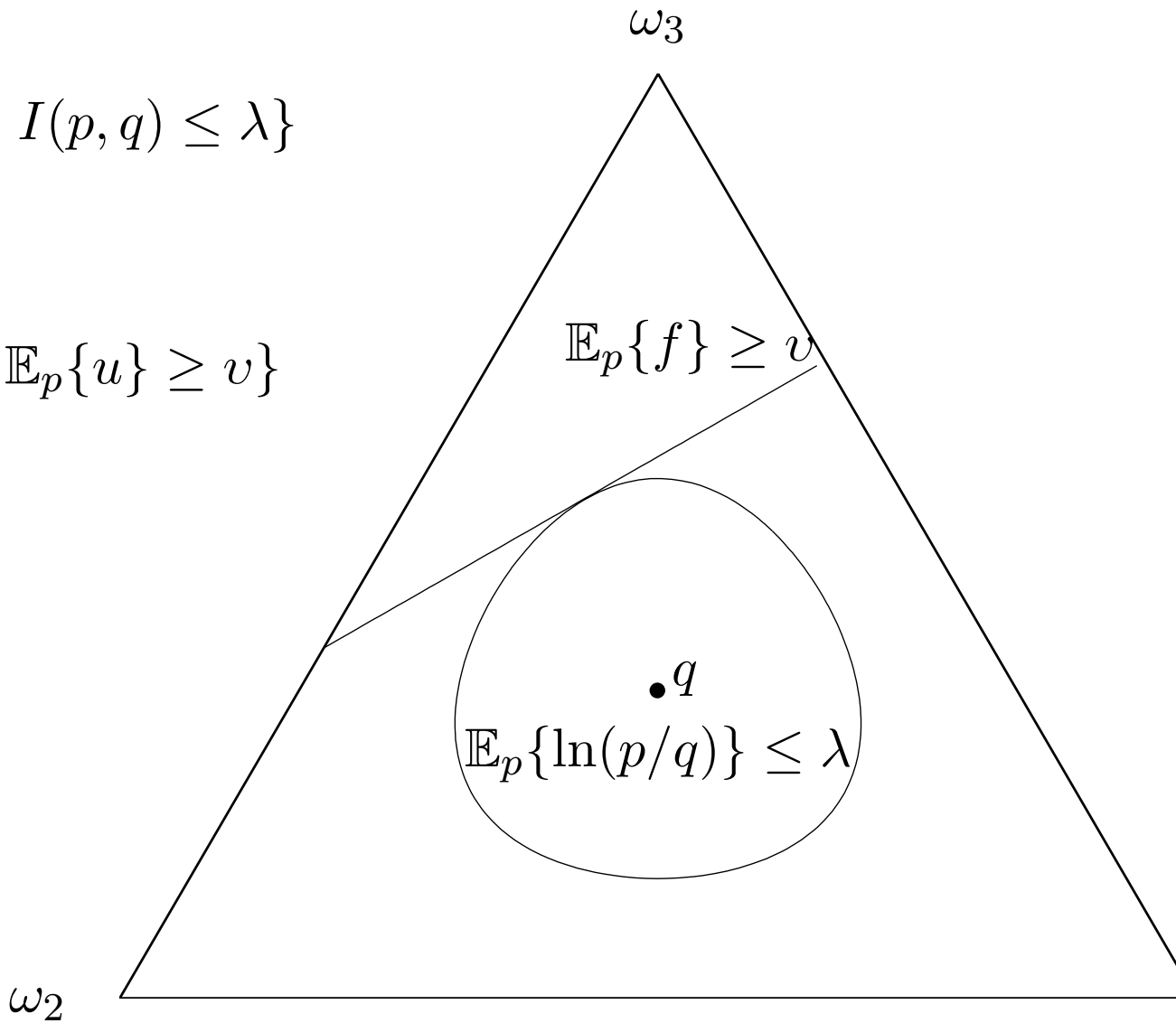
  $\lambda(V) := \inf\{I(p,q) : \mathbb{E}_p\{u\} \geq \upsilon\}$



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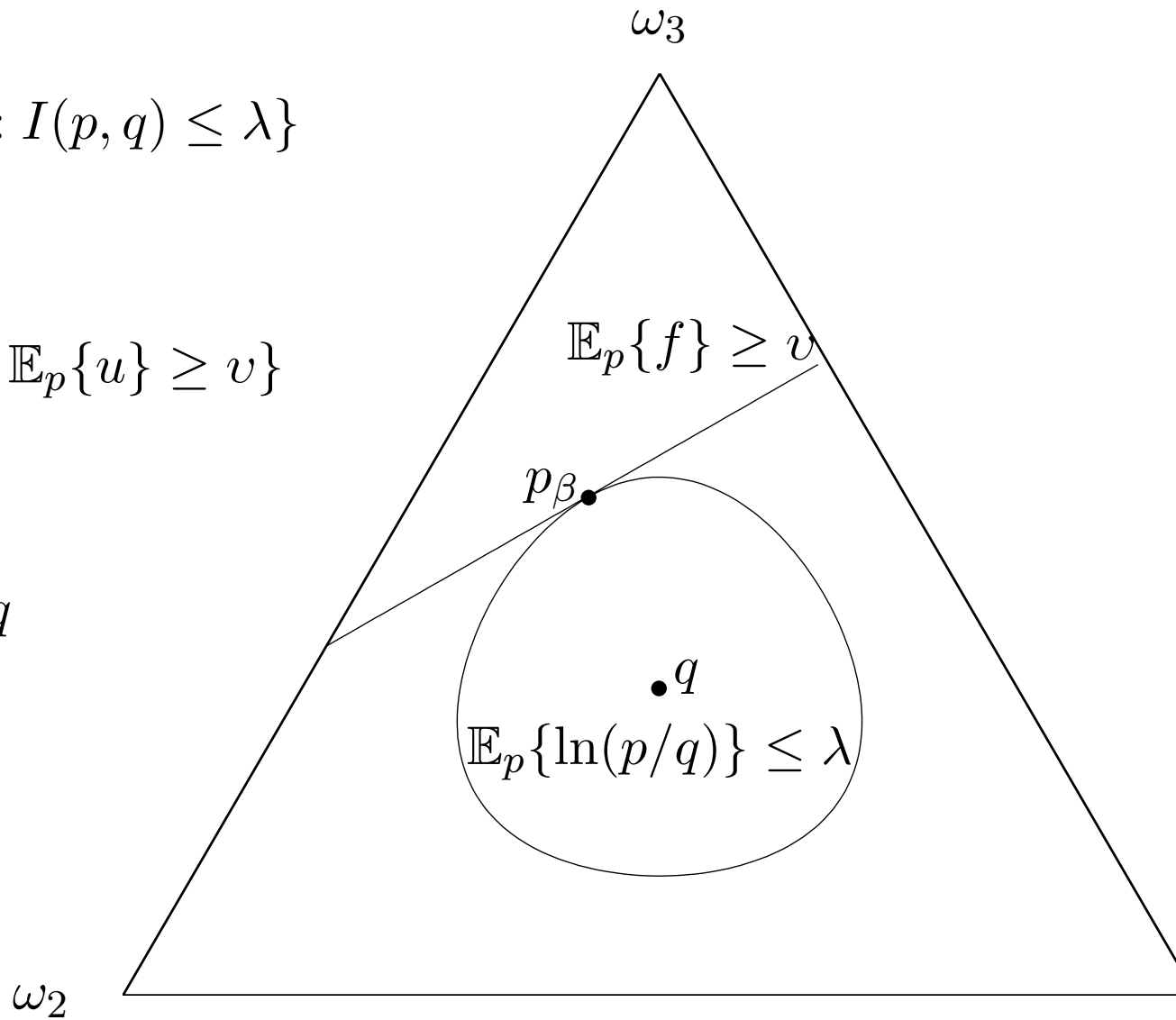
- Optimal solutions:

  $p(\beta) = e^{\beta\, u - \Psi(\beta)}\, q$



Information-geometruc view (cont)

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- Optimal solutions:

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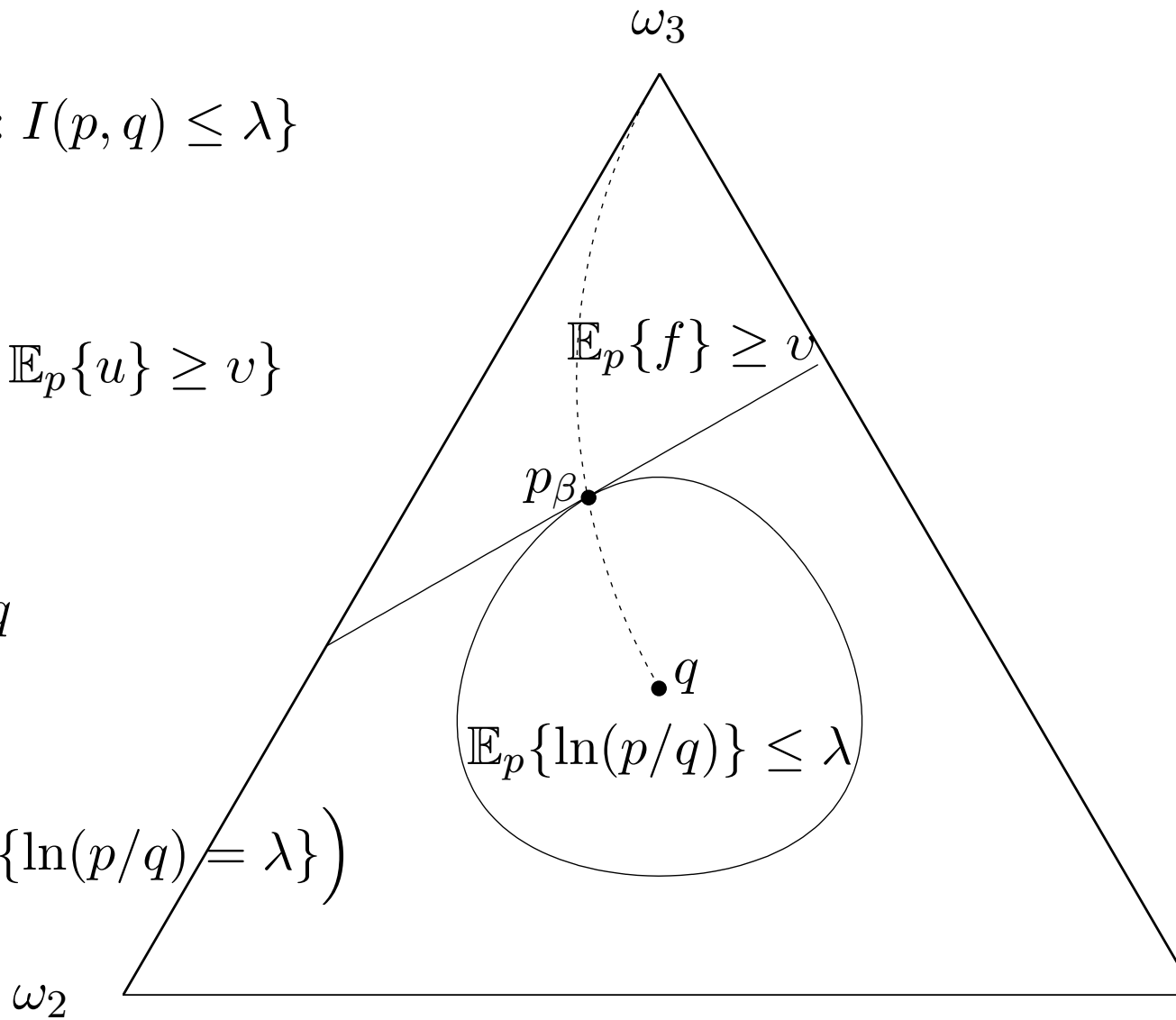
- Constraints

  $\mathbb{E}_{p(\beta)}\{u\} = v, \quad \Big(\mathbb{E}_p\{\ln(p/q) = \lambda\}\Big)$



Solution

- Lagrange function

$$K(p, \beta) = \mathbb{E}_p\{\ln(p/q)\} + \beta[\upsilon - \mathbb{E}_p\{u\}]$$

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$$\begin{aligned} \nabla_p K(p,\beta) &= \ln(p/q) + 1 - \beta u = 0 \\ \nabla_\beta K(p,\beta) &= \upsilon - \mathbb{E}_p\{u\} = 0 \end{aligned}$$

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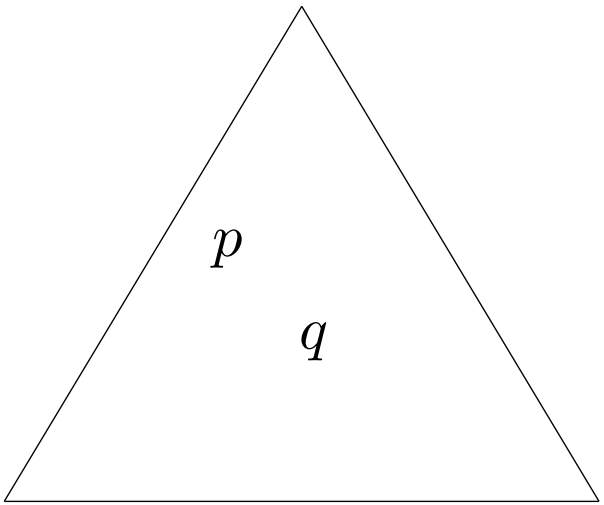
- Optimal *inverse temperature* β:

$$\beta = \frac{dK(\upsilon)}{d\upsilon} \quad \text{or} \quad \beta^{-1} = \frac{dV(\lambda)}{d\lambda}$$

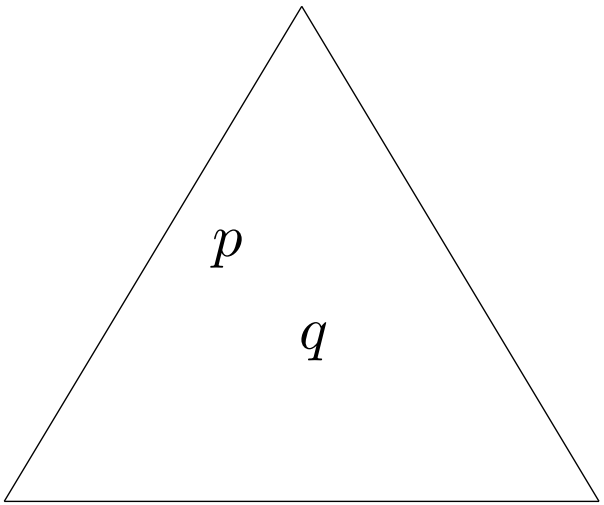
Variational problems for composite systems



$\mathcal{P}(X)$

type I : Find optimal $p \in \mathcal{P}(X)$

Variational problems for composite systems



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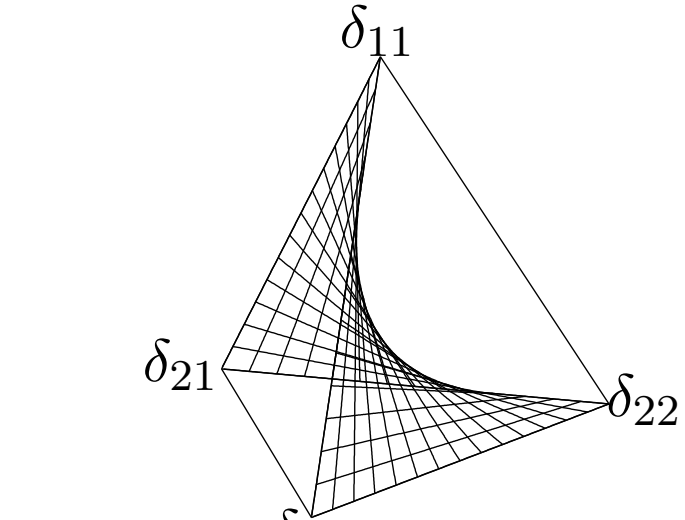

$\mathcal{P}(X \otimes Z)$

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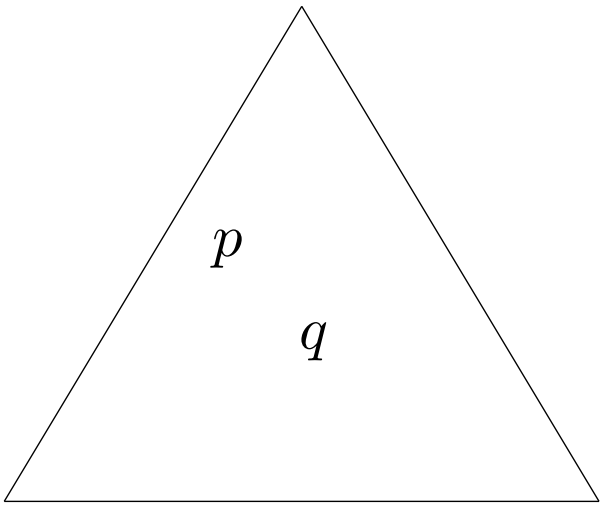
Variational problems for composite systems



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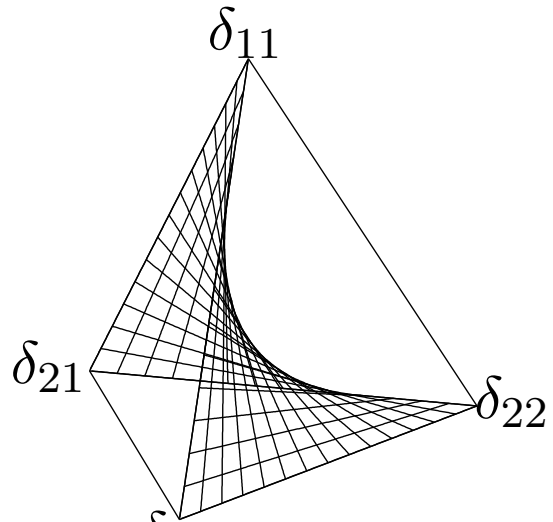

$\mathcal{P}(X \otimes Z)$

type I : Find optimal $p \in \mathcal{P}(X)$

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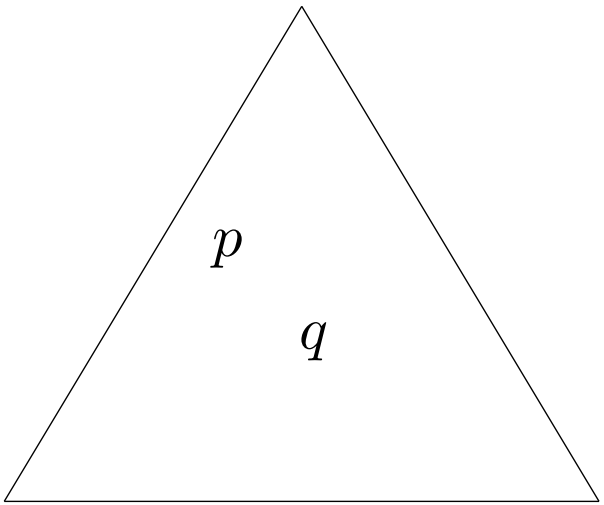
Variational problems for composite systems



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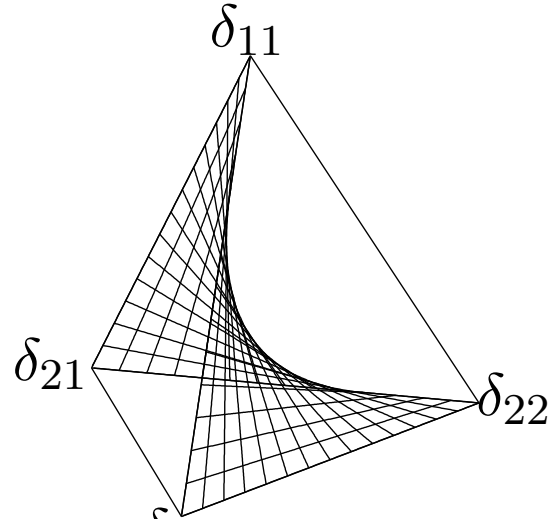

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Third variational problem

- Linear programming problem:

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- Information:

$$I(X,Z) = H(Z) - H(Z \mid X) \leq H(Z) \leq \ln |Z|$$

Optimal transition kernels

- Optimal solutions achieving $V(\lambda)$ have exponential form, such as:

$$P(z \mid x) = \frac{P(z)\, e^{\beta u(x,z)}}{\sum_z P(z)\, e^{\beta u(x,z)}}$$

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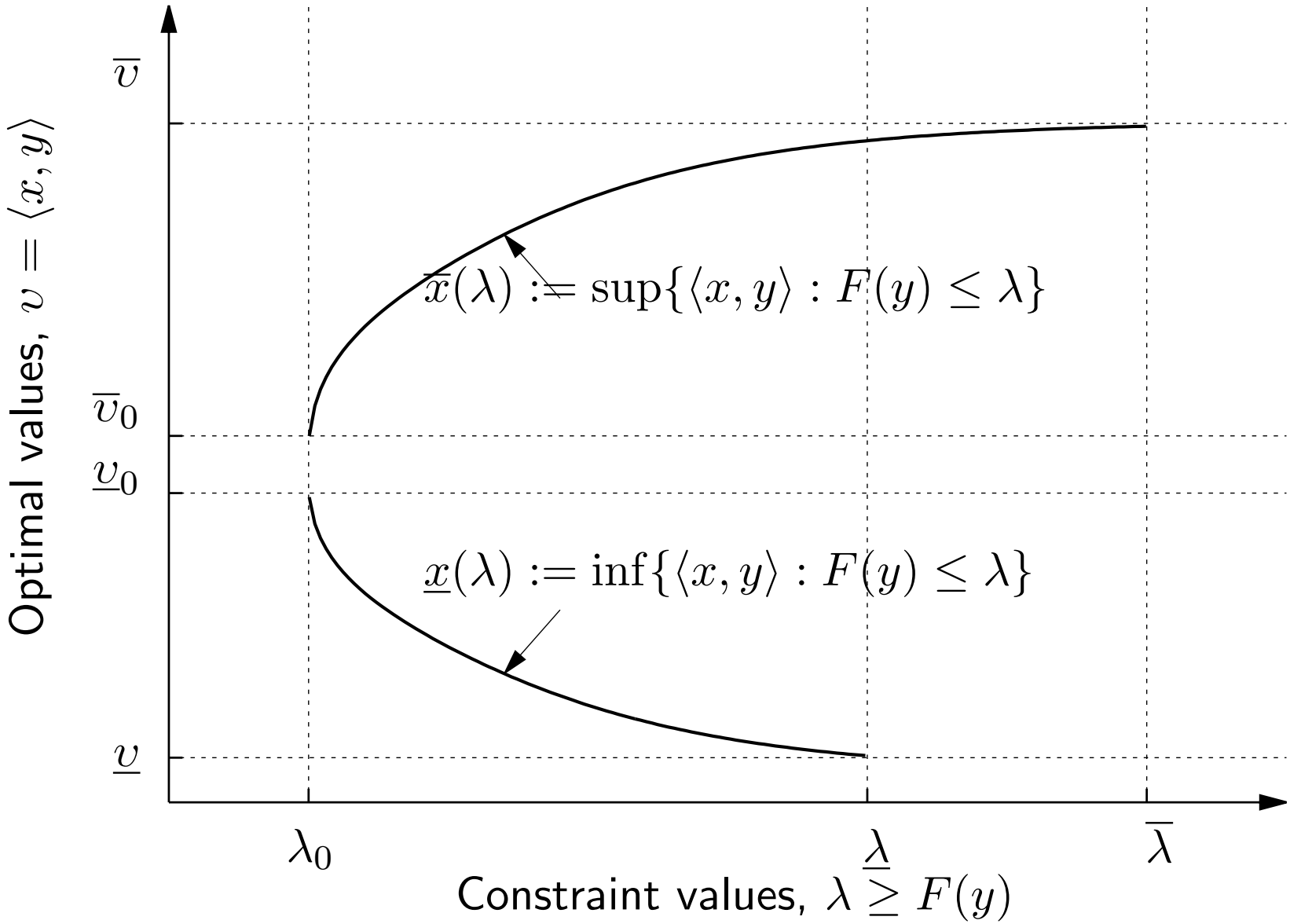
- One can show that temperature β^{-1} is the slope of $V(\lambda)$:

$$\beta^{-1} = \frac{dV(\lambda)}{d\lambda}$$

Concave and convex value functions



Value of Hartley's information

Feed-forward neural networks and VoI

Forecast and VoI

Values of Boltzmann's and Shannon's information

Solutions of VoI

McCulloch, W., & Pitts, W. (1943). A logical calculus of ideas immanent in nervous activity. *Bulletin of Mathematical Biophysics*, *5*, 115-133.

Shannon, C. E. (1948, July and October). A mathematical theory of communication. *Bell System Technical Journal*, *27*, 379–423 and 623–656.

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